The interacting boson model for exotic nuclei

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Summary. — These lectures notes give an introduction to the use of algebraic techniques for obtaining analytic eigensolutions of quantum-mechanical systems consisting of many particles in interaction. The notions of symmetry and dynamical symmetry in quantum physics are introduced and subsequently illustrated with the example of the interacting boson model of atomic nuclei. Some recent applications of this model to exotic nuclei are discussed.

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1. – Introduction

The interacting boson model (IBM) has developed steadily since its inception more than three decades ago [1] and consists by now of a family of nuclear structure models linked by common underlying assumptions concerning their microscopic foundation and by the algebraic origin of their formalism. In a wider context, the IBM represents one of the three principal approaches for the theoretical description of the atomic nucleus. The first is the independent-particle shell model which describes the nucleus in terms of independently moving particles. The second approach is the geometric model which pictures the nucleus as a dense liquid drop exhibiting vibrations around an equilibrium shape which can be spherical or deformed. The reconciliation of these two different

views of the nucleus (which both have strong empirical backing) has been one of the major endeavours of theoretical nuclear physics since the 1950s. The IBM can be viewed as a third, alternative way to understand nuclei, where the focus is on the search for symmetries in a quantal system of many interacting particles which have a geometric interpretation in the limit of large particle number. This defines two important aspects of the IBM: (i) the use of symmetry methods to obtain analytical eigensolutions of quantal many-body systems; (ii) the link between the liquid-drop and shell-model views of the nucleus.

Although the algebraic methods to be explained below have formed the basis of the IBM and its offshoots, it should be stressed that the techniques are of general interest and can be applied to any quantum-mechanical many-body system. To emphasize this aspect, the review starts with a brief reminder of the role of symmetry in quantum mechanics (sect. **2**) and of the application of group theory in quantal many-body systems (sect. **3**). As an illustration of these generic methods the example of the IBM is discussed in sect. **4**. Obviously, no attempt can be made at a complete review of the model but an outline of its symmetry structure is given, its connection with the underlying fermionic degrees of freedom is briefly outlined and its geometric interpretation in terms of the classical limit is discussed. The final two sections present examples of applications of the IBM to exotic nuclei. The first, presented in sect. **5**, gives a detailed account on how three-body interactions between the bosons can improve the description of the spectroscopic properties of certain nuclei. The second application (sect. **6**) outlines a method whereby the full IBM hamiltonian is used to obtain a simultaneous description of the binding energies and excitation spectra of a large number of nuclei in a single major shell.

2. – Symmetry in quantum mechanics

In this section it is shown how group theory can be applied to quantum mechanics. First a reminder of the concepts of symmetry and dynamical symmetry, and their consequences is given.

2˙1. *Symmetry.* – A hamiltonian $\hat{H}$ which commutes with the generators $\hat{g}_i$ that form a Lie algebra $\mathcal{G}$, that is,

$$\forall \hat{g}_i \in \mathcal{G} : [\hat{H}, \hat{g}_i] = 0, \tag{1}$$

is said to have a **symmetry** $\mathcal{G}$ or, alternatively, to be **invariant under** $\mathcal{G}$.

The determination of operators $\hat{g}_i$ that leave invariant the hamiltonian of a given physical system is central to any quantum-mechanical description. The reasons for this are profound and can be understood from the correspondence between geometrical and quantum-mechanical transformations. It can be shown that the transformations $\hat{g}_i$ with the symmetry property (1) are induced by geometrical transformations that leave unchanged the corresponding classical hamiltonian. So it is that the classical notion of a conserved quantity is transcribed in quantum mechanics in the form of the symmetry property (1) of the hamiltonian.

2˙2. *Degeneracy and state labelling.* – A well-known consequence of a symmetry is the occurrence of degeneracies in the eigenspectrum of $\hat{H}$. Given an eigenstate $|\gamma\rangle$ of $\hat{H}$ with energy E, the condition (1) implies that the states $\hat{g}_i|\gamma\rangle$ all have the same energy:

$$\hat{H}\hat{g}_i|\gamma\rangle = \hat{g}_i\hat{H}|\gamma\rangle = E\hat{g}_i|\gamma\rangle.$$

An arbitrary eigenstate of $\hat{H}$ shall be written as $|\Gamma\gamma\rangle$, where the first quantum number Γ is different for states with different energies and the second quantum number γ is needed to label degenerate eigenstates. The eigenvalues of a hamiltonian that satisfies the condition (1) depend on Γ only,

$$\hat{H}|\Gamma\gamma\rangle = E(\Gamma)|\Gamma\gamma\rangle, \tag{2}$$

and, furthermore, the transformations $\hat{g}_i$ do not admix states with different Γ:

$$\hat{g}_i|\Gamma\gamma\rangle = \sum_{\gamma'} a^{\Gamma}_{\gamma'\gamma}(i)|\Gamma\gamma'\rangle. \tag{3}$$

This simple discussion of the consequences of a hamiltonian symmetry illustrates at once the relevance of group theory in quantum mechanics. Symmetry implies degeneracy and eigenstates that are degenerate in energy provide a Hilbert space in which a matrix representation of the symmetry group can be constructed. Consequently, the representations of a given group directly determine the degeneracy structure of a hamiltonian with that symmetry.

A sufficient condition for a hamiltonian to have the symmetry property (1) is that it can be expressed in terms of Casimir operators of various orders. The eigenequation (2) then becomes

$$\left(\sum_m \kappa_m \hat{C}_m[\mathcal{G}]\right)|\Gamma\gamma\rangle = \left(\sum_m \kappa_m E_m(\Gamma)\right)|\Gamma\gamma\rangle. \tag{4}$$

In fact, the following discussion is valid for any analytic function of the various Casimir operators but mostly a linear combination is taken, as in eq. (4). The energy eigenvalues $E_m(\Gamma)$ are functions of the labels that specify the irreducible representation Γ, and are known for all classical Lie algebras.

2˙3. *Dynamical symmetry breaking.* – The concept of a dynamical symmetry can now be introduced for which (at least) two algebras $\mathcal{G}_1$ and $\mathcal{G}_2$ with $\mathcal{G}_1 \supset \mathcal{G}_2$ are needed. The eigenstates of a hamiltonian $\hat{H}$ with symmetry $\mathcal{G}_1$ are labelled as $|\Gamma_1\gamma_1\rangle$. But, since $\mathcal{G}_1 \supset \mathcal{G}_2$, a hamiltonian with $\mathcal{G}_1$ symmetry necessarily must also have a symmetry $\mathcal{G}_2$ and, consequently, its eigenstates can also be labelled as $|\Gamma_2\gamma_2\rangle$. Combination of the two properties leads to the eigenequation

$$\hat{H}|\Gamma_1\eta_{12}\Gamma_2\gamma_2\rangle = E(\Gamma_1)|\Gamma_1\eta_{12}\Gamma_2\gamma_2\rangle, \tag{5}$$

where the role of γ_1 is played by $\eta_{12}\Gamma_2\gamma_2$. In eq. (5) the irreducible representation $[\Gamma_2]$ may occur more than once in $[\Gamma_1]$, and hence an additional quantum number η_{12} is needed to uniquely label the states. Because of $\mathcal{G}_1$ symmetry, eigenvalues of $\hat{H}$ depend on Γ_1 only.

In many examples in physics, the condition of $\mathcal{G}_1$ symmetry is too strong and a *possible* breaking of the $\mathcal{G}_1$ symmetry can be imposed via the hamiltonian

$$\hat{H}' = \sum_{m_1} \kappa_{m_1} \hat{C}_{m_1}[\mathcal{G}_1] + \sum_{m_2} \kappa_{m_2} \hat{C}_{m_2}[\mathcal{G}_2], \tag{6}$$

which consists of a combination of Casimir operators of $\mathcal{G}_1$ *and* $\mathcal{G}_2$. The symmetry properties of the hamiltonian $\hat{H}'$ are now as follows. Since $[\hat{H}', \hat{g}_i] = 0$ for $\hat{g}_i \in \mathcal{G}_2$, $\hat{H}'$ is invariant under $\mathcal{G}_2$. The hamiltonian $\hat{H}'$, since it contains $\hat{C}_{m_2}[\mathcal{G}_2]$, does not commute, in general, with all elements of $\mathcal{G}_1$ and for this reason the $\mathcal{G}_1$ symmetry is broken. Nevertheless, because $\hat{H}'$ is a combination of Casimir operators of $\mathcal{G}_1$ and $\mathcal{G}_2$, its eigenvalues can be obtained in closed form:

$$\begin{aligned} &\left(\sum_{m_1} \kappa_{m_1} \hat{C}_{m_1}[\mathcal{G}_1] + \sum_{m_2} \kappa_{m_2} \hat{C}_{m_2}[\mathcal{G}_2] \right) |\Gamma_1 \eta_{12} \Gamma_2 \gamma_2\rangle \\ &\quad = \left(\sum_{m_1} \kappa_{m_1} E_{m_1}(\Gamma_1) + \sum_{m_2} \kappa_{m_2} E_{m_2}(\Gamma_2) \right) |\Gamma_1 \eta_{12} \Gamma_2 \gamma_2\rangle. \end{aligned} \tag{7}$$

The conclusion is thus that, although $\hat{H}'$ is not invariant under $\mathcal{G}_1$, its eigenstates are the same as those of $\hat{H}$ in eq. (5). The hamiltonian $\hat{H}'$ is said to have $\mathcal{G}_1$ as a **dynamical symmetry**. The essential feature is that, although the eigenvalues of $\hat{H}'$ depend on Γ_1 *and* Γ_2 (and hence $\mathcal{G}_1$ is not a symmetry), the eigenstates do not change during the breaking of the $\mathcal{G}_1$ symmetry: the dynamical symmetry breaking splits but does not admix the eigenstates. A convenient way of summarizing the symmetry character of $\hat{H}'$ and the ensuing classification of its eigenstates is as follows:

$$\begin{array}{ccc} \mathcal{G}_1 & \supset & \mathcal{G}_2 \\ \downarrow & & \downarrow \\ \Gamma_1 & & \eta_{12}\Gamma_2 \end{array} . \tag{8}$$

This equation indicates the larger algebra $\mathcal{G}_1$ (sometimes referred to as the **dynamical algebra** or **spectrum generating algebra**) and the symmetry algebra $\mathcal{G}_2$, together with their associated labels with possible multiplicities.

3. – Dynamical symmetries in quantal many-body systems

So far the discussion of symmetries has been couched in general terms leading to results that are applicable to any quantum-mechanical system. We shall now be somewhat

more specific and show how the concept of dynamical symmetry can be applied systematically to find analytical eigensolutions for a system of interacting bosons and/or fermions. As the results are most conveniently discussed in a second-quantization formalism, first a brief reminder of some essential formulas is given.

3˙1. *Many-particle states in second quantization.* – In general, particle creation and annihilation operators shall be denoted as $c_i^\dagger$ and c_i, respectively. (Note that, as these are operators, consistency with earlier notational conventions would require the notation $\hat{c}_i^\dagger$ and $\hat{c}_i$; the hats are omitted for notational simplicity.) The index i comprises the complete quantum-mechanical labelling of a single-particle state. In many applications i coincides with the labels of a stationary quantum state for a single particle in which case $c_i^\dagger$ creates a particle in that stationary state. The index i may include intrinsic quantum numbers such as spin, isospin, colour etc.

The particles are either fermions or bosons, for which the notations $c \equiv a$ and $c \equiv b$, respectively, shall be reserved. They obey different statistics, of Fermi–Dirac and of Bose–Einstein, respectively, which in second quantization is imposed through the (anti)commutation properties of creation and annihilation operators:

$$\{a_i, a_j^\dagger\} = \delta_{ij}, \qquad \{a_i^\dagger, a_j^\dagger\} = \{a_i, a_j\} = 0, \tag{9}$$

and

$$[b_i, b_j^\dagger] = \delta_{ij}, \qquad [b_i^\dagger, b_j^\dagger] = [b_i, b_j] = 0. \tag{10}$$

Introducing the notation

$$[\hat{u}, \hat{v}\}_q \equiv \hat{u}\hat{v} - (-)^q \hat{v}\hat{u}, \tag{11}$$

with $q = 0$ for bosons and $q = 1$ for fermions, one can express these relations as

$$[c_i, c_j^\dagger\}_q = \delta_{ij}, \qquad [c_i^\dagger, c_j^\dagger\}_q = [c_i, c_j\}_q = 0. \tag{12}$$

A many-particle state can be written as

$$|\bar{n}\rangle \equiv \prod_i \frac{(c_i^\dagger)^{n_i}}{\sqrt{n_i!}} |\mathrm{o}\rangle,$$

where $|\mathrm{o}\rangle$ is the vacuum state which satisfies

$$\forall i : c_i|\mathrm{o}\rangle = 0. \tag{13}$$

A many-particle state is thus completely determined by specifying the number of particles n_i in each quantum state i, and the (possibly infinite) set of numbers n_i is collectively denoted as $\bar{n}$. For fermions only $n_i = 0$ and $n_i = 1$ are allowed (since $a_i^\dagger a_i^\dagger|\mathrm{o}\rangle = -a_i^\dagger a_i^\dagger|\mathrm{o}\rangle = 0$) but for bosons no restrictions on n_i exist.

As an example of the application of the second-quantization formalism, we prove that $c_j^\dagger c_j$ is an operator that counts the number of particles in state j. For bosons, its action on a many-particle state can be worked out from the commutation relations (10) and the vacuum property (13):

$$b_j^\dagger b_j|\bar{n}\rangle = \prod_{i\neq j}\frac{(b_i^\dagger)^{n_i}}{\sqrt{n_i!}} b_j^\dagger b_j \frac{(b_j^\dagger)^{n_j}}{\sqrt{n_j!}}|\mathrm{o}\rangle = \prod_{i\neq j}\frac{(b_i^\dagger)^{n_i}}{\sqrt{n_i!}} b_j^\dagger \frac{1}{\sqrt{n_j!}}[b_j,(b_j^\dagger)^{n_j}]|\mathrm{o}\rangle.$$

Since, for arbitrary operators $\hat{u}$, $\hat{v}$ and $\hat{w}$,

$$[\hat{u},\hat{v}\hat{w}] = \hat{v}[\hat{u},\hat{w}] + [\hat{u},\hat{v}]\hat{w},$$

one finds

$$[b_j,(b_j^\dagger)^{n_j}] = (b_j^\dagger)^{n_j-1} + [b_j,(b_j^\dagger)^{n_j-1}]b_j^\dagger = \cdots = n_j(b_j^\dagger)^{n_j-1},$$

and thus

$$b_j^\dagger b_j|\bar{n}\rangle = n_j|\bar{n}\rangle.$$

The equivalent property for fermions can be shown simply by noting that either $n_j = 0$ or $n_j = 1$, in which case $a_j^\dagger a_j$ gives zero or one, respectively. Hence, in summary we have

$$c_j^\dagger c_j|\bar{n}\rangle = n_j|\bar{n}\rangle.$$

and, because of this property, $c_j^\dagger c_j$ is called a number operator.

3˙2. *Particle-number conserving spectrum generating algebras*. – The determination of the properties of a quantal system of N interacting particles requires the solution of the eigenvalue equation associated with the hamiltonian

$$\hat{H} = \sum_i \epsilon_i c_i^\dagger c_i + \sum_{ijkl} v_{ijkl} c_i^\dagger c_j^\dagger c_k c_l + \cdots, \tag{14}$$

containing one-body terms ϵ_i, two-body interactions v_{ijkl} and so on; higher-order interactions can be included in the expansion, if needed. The hamiltonian (14) satisfies the requirement of particle-number conservation; the case that does not conserve particle number is discussed in the next subsection.

Note that the assumed diagonality of the one-body term does not make the hamiltonian (14) less general. In fact, a non-diagonal one-body term

$$\sum_{ij}\epsilon_{ij}c_i^\dagger c_j,$$

can always be brought in diagonal form through a unitary transformation of the form

$$c_j = \sum_s u_{js} c'_s, \qquad c_i^\dagger = \sum_r u_{ir}^* c_r'^\dagger,$$

which yields

$$\sum_{ij} \epsilon_{ij} c_i^\dagger c_j = \sum_{rs} \left(\sum_{ij} u_{ir}^* \epsilon_{ij} u_{js} \right) c_r'^\dagger c'_s \equiv \sum_r \epsilon'_r c_r'^\dagger c'_r,$$

where the last equality is obtained by choosing the unitary transformation such that it diagonalizes ϵ_{ij}:

$$\sum_{ij} u_{ir}^* \epsilon_{ij} u_{js} = \epsilon'_r \delta_{rs}.$$

Since the transformation is unitary, its inverse is also and the new single-particle states still obey the same (anti)commutation relations (12):

$$[c'_r, c_s'^\dagger\}_q = \sum_{ij} u_{ri}^{-1} u_{sj}^{-1*} [c_i, c_j^\dagger\}_q = \sum_i u_{ri}^{-1} u_{si}^{-1*} = \delta_{rs}.$$

With use of the property [see eqs. (11,12)]

$$c_j^\dagger c_k = (-)^q c_k c_j^\dagger - (-)^q \delta_{jk},$$

the hamiltonian (14) can be written in a different form as

$$\hat{H} = \sum_{il} \left(\epsilon_i \delta_{il} - (-)^q \sum_j v_{ijkl} \right) \hat{u}_{il} + (-)^q \sum_{ijkl} \upsilon_{ijkl} \hat{u}_{ik} \hat{u}_{jl} + \cdots, \tag{15}$$

where the notation $\hat{u}_{ij} \equiv c_i^\dagger c_j$ is introduced. The reason for doing so becomes clear when the commutator of the $\hat{u}_{ij}$ operators is considered:

$$[\hat{u}_{ij}, \hat{u}_{kl}] = c_i^\dagger c_k^\dagger [c_j, c_l] + c_i^\dagger [c_j, c_k^\dagger] c_l - c_k^\dagger [c_l, c_i^\dagger] c_j + [c_i^\dagger, c_k^\dagger] c_l c_j,$$

which, because of the identity

$$[\hat{u}, \hat{v}] = [\hat{u}, \hat{v}\} - (1 - (-)^q)\, \hat{v}\hat{u},$$

can be brought into the form

$$[\hat{u}_{ij}, \hat{u}_{kl}] = \hat{u}_{il}\delta_{jk} - \hat{u}_{kj}\delta_{il} - (1 - (-)^q) \left[c_i^\dagger c_k^\dagger c_l c_j + c_i^\dagger c_k^\dagger c_j c_l \right].$$

The last term on the right-hand-side of this equation is zero for bosons (when $q = 0$) as it is for fermions since in that case the expression between square brackets vanishes. This shows that

$$[\hat{u}_{ij}, \hat{u}_{kl}] = \hat{u}_{il}\delta_{jk} - \hat{u}_{kj}\delta_{il} \tag{16}$$

is valid for a boson as well as a fermion realization of the $\hat{u}_{ij}$ and that these operators in both cases satisfy the commutation relations of the unitary algebra U(n) where n is the dimensionality of the single-particle space.

The equivalent form (15) shows that the solution of the eigenvalue problem for N particles associated with the hamiltonian (14) requires the diagonalization of $\hat{H}$ in the symmetric representation $[N]$ of U(n) in case of bosons or in its antisymmetric representation $[1^N]$ in case of fermions. This, for a general hamiltonian, is a numerical problem which quickly becomes intractable with increasing numbers of particles N or increasing single-particle space n. A strategy for solving *particular classes* of the many-body hamiltonian (14) can be obtained by considering the algebra U(n) as a spectrum generating or dynamical algebra $\mathcal{G}_{\rm dyn}$ on which a dynamical symmetry breaking is applied. The generalization of the procedure of sect. **2**˙**3** is straightforward and starts from a chain of nested algebras

$$\mathcal{G}_1 \equiv \mathcal{G}_{\rm dyn} \supset \mathcal{G}_2 \supset \cdots \supset \mathcal{G}_s \equiv \mathcal{G}_{\rm sym}, \tag{17}$$

where the last algebra $\mathcal{G}_s$ in the chain is the symmetry algebra of the problem. To appreciate the relevance of this classification in connection with the many-body problem (14), one associates with the chain (17) the hamiltonian

$$\hat{H} = \sum_{r=1}^{s} \sum_{m} \kappa_{rm} \hat{C}_m[\mathcal{G}_r], \tag{18}$$

which represents a direct generalization of eq. (6) and where κ_{rm} are arbitrary coefficients. The operators in the hamiltonian (18) satisfy

$$\forall m, m', r, r' : [\hat{C}_m[\mathcal{G}_r], \hat{C}_{m'}[\mathcal{G}_{r'}]] = 0.$$

This property is evident from the fact that all elements of $\mathcal{G}_r$ are in $\mathcal{G}_{r'}$ or *vice versa*. Hence, the hamiltonian (18) is written as a sum of commuting operators and as a result its eigenstates are labelled by the quantum numbers associated with these operators. Note that the condition of the *nesting* of the algebras is crucial for constructing a set of commuting operators and hence for obtaining an analytical solution. Since the Casimir operators can be expressed in terms of the operators $\hat{u}_{ij}$, the expansion (18) can, in principle, be rewritten in the form (15) with the order of the interactions determined by the maximal order m of the invariants.

To summarize these results, the hamiltonian (18)—which can be obtained from the general hamiltonian (14) for specific coefficients ϵ_i, $v_{ijkl}\ldots$—can be solved analytically.

Its eigenstates do not depend on the coefficients κ_{rm} and are labelled by

$$\begin{array}{ccccccc} \mathcal{G}_1 & \supset & \mathcal{G}_2 & \supset & \cdots & \supset & \mathcal{G}_s \\ \downarrow & & \downarrow & & & & \downarrow \\ \Gamma_1 & & \eta_{12}\Gamma_2 & & & & \eta_{s-1,s}\Gamma_s \end{array} . \tag{19}$$

Its eigenvalues are given in closed form as

$$\hat{H}|\Gamma_1\eta_{12}\Gamma_2\ldots\eta_{s-1,s}\Gamma_s\rangle = \sum_{r=1}^{s}\sum_{m}\kappa_{rm}E_m(\Gamma_r)|\Gamma_1\eta_{12}\Gamma_2\ldots\eta_{s-1,s}\Gamma_s\rangle, \tag{20}$$

where $E_m(\Gamma_r)$ are known functions introduced in sect. **2**˙2.

Thus a generic scheme is established for finding analytically solvable hamiltonians (14): it requires an enumeration of all nested chains of the type (17) which is a purely algebraic problem. The symmetry of the dynamical algebra $\mathcal{G}_{\rm dyn}$ is broken dynamically and the only remaining symmetry is $\mathcal{G}_{\rm sym}$ which is the true symmetry of the problem.

This approach to find analytical eigensolutions for a system of interacting bosons and/or fermions has received prominence with the work of Arima and Iachello [1, 2] where the dynamical algebra $\mathcal{G}_{\rm dyn}$ is U(6) and the symmetry algebra $\mathcal{G}_{\rm sym}$ is the rotational algebra SO(3), as will be described in detail in sect. **4**. It should not be forgotten, however, that these symmetry methods had been used before in different models in physics. The Isobaric Multiplet Mass Equation [3] as well as the Gell-Mann–Okubo mass formula [4, 5] can be viewed as examples of symmetry breaking of the type (19). A beautiful example in nuclear physics is Elliott's rotational SU(3) model [6] in which Wigner's supermultiplet [7] degeneracy associated with SU(4) is lifted dynamically by the quadrupole interaction. The technique has also been applied to the nuclear shell model. Based on the methods developed for the IBM, a systematic procedure for constructing analytically solvable Hamiltonians was devised by Ginocchio [8], drawing on earlier ideas by Hecht *et al.* [9] and using a method which resembles that of pseudo spin [10, 11]. The theory was later developed under the name of fermion dynamical symmetry model [12].

3˙3. *Particle-number non-conserving dynamical algebras.* – The hamiltonians constructed from the unitary generators u_{ij} necessarily conserve particle number since that is so for the generators themselves. In many cases (involving, *e.g.*, virtual particles, effective phonon-like excitations...) no particle-number conservation can be imposed and a more general formalism is required. Another justification for such generalizations is that the strategy outlined in sect. **3**˙2 has the drawback that the dynamical algebra $\mathcal{G}_{\rm dyn}$ can become very large (due a large single-particle space combined with possible intrinsic quantum numbers such as spin and isospin) which makes the analysis of the group-theoretical reduction (17), and the associated labelling (19) in particular, too difficult to be of practical use. In some cases the following, more economical, procedure is called for.

In addition to the unitary generators u_{ij}, also the operators $\hat{s}_{ij} \equiv c_i c_j$ and $\hat{s}^\dagger_{ij} \equiv c^\dagger_i c^\dagger_j$ are considered. [Note that this notation implies $\hat{s}^\dagger_{ij} = (\hat{s}_{ji})^\dagger$.] We now show that the set

of operators $\hat{s}_{ij}$ and $\hat{s}^\dagger_{ij}$, added to $\hat{u}'_{ij} \equiv \hat{u}_{ij} + \frac{1}{2}(-)^q\delta_{ij}$, forms a closed algebra. Since the operators $\hat{u}_{ij}$ and $\hat{u}'_{ij}$ differ by a constant only, they satisfy the same commutation relations (16), and in the same way it can be shown that

$$[\hat{u}'_{ij}, \hat{s}_{kl}] = -\hat{s}_{kj}\delta_{il} - \hat{s}_{jl}\delta_{ik}, \qquad [\hat{u}'_{ij}, \hat{s}^\dagger_{kl}] = \hat{s}^\dagger_{il}\delta_{jk} + \hat{s}^\dagger_{ki}\delta_{jl}.$$

The commutator of $\hat{s}_{ij}$ with $\hat{s}^\dagger_{kl}$ deserves a more detailed consideration:

$$\begin{aligned}[\hat{s}_{ij}, \hat{s}^\dagger_{kl}] = {} & c_i c^\dagger_k \delta_{jl} + c_i c^\dagger_l \delta_{jk} + c^\dagger_k c_j \delta_{il} + c^\dagger_l c_j \delta_{ik} \\ & - (1 - (-)^q) \left[c_i c^\dagger_k c^\dagger_l c_j + c_i c^\dagger_k c_j c^\dagger_l + c^\dagger_k c^\dagger_l c_i c_j + c^\dagger_k c_i c^\dagger_l c_j \right].\end{aligned}$$

The term between square brackets can be worked out by assuming that the c's are fermions since for bosons the factor $(1 - (-)^q)$ in front is zero. This gives

$$\begin{aligned}[\hat{s}_{ij}, \hat{s}^\dagger_{kl}] &= (-)^q c^\dagger_k c_i \delta_{jl} + (-)^q c^\dagger_l c_i \delta_{jk} + c^\dagger_k c_j \delta_{il} + c^\dagger_l c_j \delta_{ik} + \delta_{il}\delta_{jk} + \delta_{ik}\delta_{jl} \\ &\quad - (1 - (-)^q) \left[-c^\dagger_k c_i \delta_{jl} + c^\dagger_k c_j \delta_{il} + \delta_{ik}\delta_{jl} \right] \\ &= (-)^q \left(\hat{u}_{li}\delta_{jk} + \hat{u}_{kj}\delta_{il} + \delta_{ik}\delta_{jl} \right) + \left(\hat{u}_{ki}\delta_{jl} + \hat{u}_{lj}\delta_{ik} + \delta_{il}\delta_{jk} \right),\end{aligned}$$

and leads to the following result, valid for both fermions and bosons:

$$[\hat{s}_{ij}, \hat{s}^\dagger_{kl}] = (-)^q \hat{u}'_{li}\delta_{jk} + (-)^q \hat{u}'_{kj}\delta_{il} + \hat{u}'_{ki}\delta_{jl} + \hat{u}'_{lj}\delta_{ik}.$$

The modification $\hat{u}_{ij} \rightarrow \hat{u}'_{ij}$ is thus necessary to ensure closure of the commutator. The set $\{\hat{u}'_{ij}, \hat{s}_{ij}, \hat{s}^\dagger_{ij}\}$ contains $n(2n+1)$ or $n(2n-1)$ independent generators for bosons or fermions, respectively. From dimensionality (but also from the commutation relations) it can be inferred that the respective Lie algebras are Sp($2n$) and SO($2n$).

It is clear that these algebras can be used to construct number non-conserving hamiltonians. However, the addition of the pair creation and annihilation operators enlarges rather than diminishes the dimension of the dynamical algebra and does not lead to a simplification of the algebraic structure of the problem. The latter can be achieved by considering specific linear combinations

$$\hat{U}(\bar{\alpha}) \equiv \sum_{ij} \alpha_{ij}\hat{u}_{ij}, \quad \hat{S}_+(\bar{\beta}) \equiv \sum_{ij} \beta_{ij}\hat{s}^\dagger_{ij}, \quad \hat{S}_-(\bar{\beta}) \equiv \left(\hat{S}_+(\bar{\beta}) \right)^\dagger,$$

where the coefficients α_{ij} and β_{ij} are chosen to ensure closure to a *subalgebra* of either Sp($2n$) or SO($2n$).

This procedure will not be formally developed further here. We note that, among the nuclear models, several examples are encountered that illustrate the approach, such as the SU(2) quasi-spin algebra [13] or the SO(8) algebra of neutron–proton pairing [14].

4. – The interacting boson model

In this section an introduction to the IBM is given with particular emphasis on the version of the model which includes higher-order interactions between the bosons. A full account of the IBM is given by Iachello and Arima [2].

Before turning to a detailed discussion of the IBM, it is worthwhile to summarize the philosophy of the model as well as the most important results obtained with it. In the IBM the nucleus is described in terms of interacting s and d bosons (more about its justification in sect. **4**˙4). For such a system three different classes of analytical solutions or **limits** exist: the vibrational U(5) limit [15], the rotational SU(3) limit [16] and the γ-unstable SO(6) limit [17]. While at the time of arrival of the IBM (1975), the vibrational and rotational limits were well-recognized features in the nuclear landscape, this was not the case for the third limit. The SO(6) limit still stands out as an excellent example of the value and power of symmetry methods; its origins were purely algebraic, but its structure was later found to resemble that of a γ-unstable rotor. Its predictions were found to correspond closely to the empirical structure of some Pt nuclei [18]. Since that early work, it has become increasingly evident that the SO(6) symmetry in fact represents the third commonly occurring class of nuclei, which have been identified in several regions, most notably around $A = 130$ [19].

A second landmark contribution of the IBM to nuclear physics is **supersymmetry**. Its starting point is the extension of the IBM to odd-mass nuclei, achieved by considering, in addition to the bosons, a single fermion [20]. The resulting interacting boson–fermion model (IBFM) lends itself equally well as the IBM to a study based on symmetry considerations whereby certain classes of model hamiltonians can be solved analytically [21]. A particularly attractive feature is the conceptual similarity in the description of even–even and odd-mass nuclei. While the spectrum generating algebra of the IBM is $\mathrm{U^B}(6)$, the one of the IBFM is $\mathrm{U^B}(6) \otimes \mathrm{U^F}(\Omega)$, where Ω is the size of the single-particle space available to the fermion and the superscripts B and F are added to indicate the boson or fermion realization of the Lie algebra. The Lie algebra $\mathrm{U^B}(6) \otimes \mathrm{U^F}(\Omega)$ provides a *separate* description of even–even and odd-mass nuclei: Although the treatment is similar in both cases, no operator exists that *connects* even–even and odd-mass states. An extension, proposed by Iachello [22], considers in addition operators that transform a boson into a fermion or *vice versa*. The resulting set of operators does not any longer form a classical Lie algebra which is defined in terms of commutation relations. Instead, to define a closed algebraic structure, one needs to introduce an internal operation that corresponds to a mixture of commutation and anticommutation and the resulting algebra is called a *graded* or *super*algebra, denoted by $\mathrm{U}(6/\Omega)$. The supersymmetric generators thus induce a connection between even–even and odd-mass nuclei and lead to a *simultaneous* treatment of such pairs of nuclei. At the basis of this unified treatment is the enlargement of the dynamical algebra. This process of enlargement can be continued and from it results unified descriptions of ever higher numbers of nuclei. A further example of this mechanism is obtained if a distinction is made between neutrons and protons, both for fermions and for bosons. It then follows that a quartet of nuclei (even–even, even–odd, odd–even

and odd–odd) is connected by the supersymmetric operators and that this quartet can be described simultaneously with a single hamiltonian [23]. This approach continues to inspire the study of odd–odd nuclei to the present day, see for example [24].

A third landmark contribution of the IBM is the prediction of neutron–proton **non-symmetric** states. Given the microscopic interpretation of the bosons as correlated pairs of nucleons, a natural extension of the IBM-1 (the simplest version of the IBM) is to assume two different types of bosons, neutron and proton, giving rise to the neutron–proton interacting boson model or IBM-2 [25, 26]. The algebraic structure of IBM-2 is a product of U(6) algebras, $\mathrm{U}_\nu(6) \otimes \mathrm{U}_\pi(6)$, consisting of neutron (ν) and proton (π) generators, respectively. The most important aspect of IBM-2 is that it predicts states which are additional to those found in IBM-1 [27]. The states lowest in energy are symmetric in U(6) and are the analogues of those in IBM-1. The next class of states no longer is symmetric in U(6); they are observed experimentally [28] and seem to be a persistent feature of nuclei [29]. From a geometric analysis of non-symmetric states emerges that they correspond to linear or angular displacement oscillations in which the neutrons and protons are out of phase, in contrast to the symmetric IBM-2 states for which such oscillations are in phase. The occurrence of such states was first predicted in the context of geometric two-fluid models in vibrational [30] and deformed [31] nuclei in which they appear as neutron–proton counter oscillations. The IBM-2 thus confirms these geometric descriptions but at the same time generalizes them to *all* nuclei, not only spherical and deformed, but γ unstable and transitional as well. It is precisely in the latter case that the example of ^{94}Mo has been shown to agree to a remarkable extent with the predictions of the SO(6) limit of the IBM-2 [32].

4˙1. *Hamiltonian.* – The building blocks of the IBM are s and d bosons with angular momenta $\ell = 0$ and $\ell = 2$. A nucleus is characterized by a constant total number of bosons N which equals half the number of valence nucleons (particles or holes, whichever is smaller). In these notes no distinction is made between neutron and proton bosons, an approximation which is known as IBM-1.

Since the hamiltonian of the IBM-1 conserves the total number of bosons, it can be written in terms of the 36 operators $b^\dagger_{\ell m} b_{\ell' m'}$ where $b^\dagger_{\ell m}$ ($b_{\ell m}$) creates (annihilates) a boson with angular momentum ℓ and z projection m. According to eq. (16) this set of 36 operators generates the Lie algebra U(6). A hamiltonian that conserves the total number of bosons is of the generic form

$$\hat{H} = E_0 + \hat{H}_1 + \hat{H}_2 + \hat{H}_3 + \cdots, \tag{21}$$

where the index refers to the order of the interaction in the generators of U(6). The first term E_0 is a constant which represents the binding energy of the core. The second term is the one-body part

$$\hat{H}_1 = \epsilon_s [s^\dagger \times \tilde{s}]^{(0)} + \epsilon_d \sqrt{5} [d^\dagger \times \tilde{d}]^{(0)} \equiv \epsilon_s \hat{n}_s + \epsilon_d \hat{n}_d, \tag{22}$$

TABLE I. – *Enumeration of n-body interactions in IBM-1 for $n \leq 3$.*

Order	Number of interactions		
	total	type I[a]	type II[b]
$n = 0$	1	1	0
$n = 1$	2	1	1
$n = 2$	7	2	5
$n = 3$	17	7	10

[a]Interaction energy is constant for all states with the same N.
[b]Interaction energy varies from state to state.

where $\times$ refers to coupling in angular momentum (shown as an upperscript in round brackets), $\tilde{b}_{\ell m} \equiv (-)^{\ell-m} b_{\ell,-m}$ and the coefficients ϵ_s and ϵ_d are the energies of the s and d bosons. The third term in the hamiltonian (21) represents the two-body interaction

$$\hat{H}_2 = \sum_{\ell_1 \leq \ell_2, \ell_1' \leq \ell_2', L} \tilde{v}^L_{\ell_1 \ell_2 \ell_1' \ell_2'} [[b^\dagger_{\ell_1} \times b^\dagger_{\ell_2}]^{(L)} \times [\tilde{b}_{\ell_2'} \times \tilde{b}_{\ell_1'}]^{(L)}]^{(0)}_0, \tag{23}$$

where the coefficients $\tilde{v}$ are related to the interaction matrix elements between normalized two-boson states,

$$\langle \ell_1 \ell_2; LM | \hat{H}_2 | \ell_1' \ell_2'; LM \rangle = \sqrt{\frac{(1+\delta_{\ell_1 \ell_2})(1+\delta_{\ell_1' \ell_2'})}{2L+1}} \tilde{v}^L_{\ell_1 \ell_2 \ell_1' \ell_2'}.$$

Since the bosons are necessarily symmetrically coupled, allowed two-boson states are s^2 ($L = 0$), sd ($L = 2$) and d^2 ($L = 0, 2, 4$). Since for n states with a given angular momentum one has $n(n+1)/2$ interactions, seven independent two-body interactions v are found: three for $L = 0$, three for $L = 2$ and one for $L = 4$.

This analysis can be extended to higher-order interactions. One may consider, for example, the three-body interactions $\langle \ell_1 \ell_2 \ell_3; LM | \hat{H}_3 | \ell_1' \ell_2' \ell_3'; LM \rangle$. The allowed three-boson states are s^3 ($L = 0$), $s^2 d$ ($L = 2$), sd^2 ($L = 0, 2, 4$) and d^3 ($L = 0, 2, 3, 4, 6$), leading to $6 + 6 + 1 + 3 + 1 = 17$ independent three-body interactions for $L = 0, 2, 3, 4, 6$, respectively. Note that any three-boson state $s^i d^{3-i}$ is fully characterized by its angular momentum L; this is no longer the case for higher boson numbers when additional labels must be introduced.

The number of possible interactions at each order n is summarized in table I for up to $n = 3$. Some of these interactions contribute to the binding energy but do not influence the excitation spectrum of a nucleus. To determine the number of such interactions, one notes that the hamiltonian $\hat{N}\hat{H}_{n-1}$ for constant boson number (*i.e.*, a single nucleus) essentially reduces to the $(n-1)$-body hamiltonian $\hat{H}_{n-1}$. Consequently, of the $\mathcal{N}_n$ independent interactions of order n contained in $\hat{H}_n$, $\mathcal{N}_{n-1}$ terms of the type $\hat{N}\hat{H}_{n-1}$ must be discarded if one wishes to retain only those that influence the excitation energies.

For example, given that there is one term of order zero (*i.e.*, a constant), one of the two first-order terms (*i.e.*, the combination $\hat{N}$) does not influence the excitation spectrum. Likewise, there are two first-order terms (*i.e.*, $\hat{n}_s$ and $\hat{n}_d$) and hence two of the seven two-body interactions do not influence the excitation spectrum. This argument leads to the numbers quoted in table I.

We conclude that for fits of excitation spectra there is a single one-boson energy of relevance, as well as five two-body and ten three-body interactions. If also binding energies are included in the analysis, an additional one-boson energy can be considered as well as two two-body and seven three-body interactions. These numbers of parameters are rather high for practical applications and simplifications must be sought on the basis of physical, empirical or symmetry arguments. To the latter we now turn.

4˙2. *Dynamical symmetries.* – The characteristics of the most general IBM hamiltonian which includes up to two-body interactions and its group-theoretical properties are by now well understood [33]. Numerical procedures exist to obtain its eigensolutions but the problem can be solved analytically for particular choices of boson energies and boson–boson interactions. For an IBM hamiltonian with up to two-body interactions between the bosons, three different analytical solutions or limits exist: the **vibrational U(5)** [15], the **rotational SU(3)** [16] and the **γ-unstable SO(6)** limit [17]. They are associated with the algebraic reductions

$$\mathrm{U}(6) \supset \left\{ \begin{array}{c} \mathrm{U}(5) \supset \mathrm{SO}(5) \\ \mathrm{SU}(3) \\ \mathrm{SO}(6) \supset \mathrm{SO}(5) \end{array} \right\} \supset \mathrm{SO}(3). \tag{24}$$

The algebras appearing in the lattice (24) are subalgebras of U(6) generated by operators of the type $b^\dagger_{lm} b_{l'm'}$, the explicit form of which is listed, for example, in ref. [2]. With the subalgebras U(5), SU(3), SO(6), SO(5) and SO(3) there are associated one linear [of U(5)] and five quadratic Casimir operators. This matches the number of one- and two-body interactions quoted in the last column of table I. The total of all one- and two-body interactions can be represented by including in addition the operators $\hat{C}_1[\mathrm{U}(6)]$, $\hat{C}_2[\mathrm{U}(6)]$ and $\hat{C}_1[\mathrm{U}(6)]\hat{C}_1[\mathrm{U}(5)]$. The most general IBM hamiltonian with up to two-body interactions can thus be written in an *exactly* equivalent way with Casimir operators. Specifically, the hamiltonian reads

$$\begin{aligned} \hat{H}_{1+2} = {} & \kappa_1 \hat{C}_1[\mathrm{U}(5)] + \kappa_1' \hat{C}_2[\mathrm{U}(5)] + \kappa_2 \hat{C}_2[\mathrm{SU}(3)] \\ & + \kappa_3 \hat{C}_2[\mathrm{SO}(6)] + \kappa_4 \hat{C}_2[\mathrm{SO}(5)] + \kappa_5 \hat{C}_2[\mathrm{SO}(3)], \end{aligned} \tag{25}$$

which is just an alternative way of writing $\hat{H}_1 + \hat{H}_2$ of eqs. (22,23) if interactions are omitted that contribute to the binding energy only.

The representation (25) is much more telling when it comes to the symmetry properties of the IBM hamiltonian. If some of the coefficients κ_i vanish such that $\hat{H}_{1+2}$ contains Casimir operators of subalgebras belonging to a *single* reduction in the lattice (24), then,

according to the discussion of sect. **2**˙3, the eigenvalue problem can be solved analytically. Three classes of spectrum generating hamiltonians can thus be constructed of the form

$$\begin{aligned}
&\mathrm{U}(5): \hat{H}_{1+2} = \kappa_1 \hat{C}_1[\mathrm{U}(5)] + \kappa_1' \hat{C}_2[\mathrm{U}(5)] + \kappa_4 \hat{C}_2[\mathrm{SO}(5)] + \kappa_5 \hat{C}_2[\mathrm{SO}(3)], \\
&\mathrm{SU}(3): \hat{H}_{1+2} = \kappa_2 \hat{C}_2[\mathrm{SU}(3)] + \kappa_5 \hat{C}_2[\mathrm{SO}(3)], \\
&\mathrm{SO}(6): \hat{H}_{1+2} = \kappa_3 \hat{C}_2[\mathrm{SO}(6)] + \kappa_4 \hat{C}_2[\mathrm{SO}(5)] + \kappa_5 \hat{C}_2[\mathrm{SO}(3)].
\end{aligned} \tag{26}$$

In each of these limits the hamiltonian is written as a sum of commuting operators and, as a consequence, the quantum numbers associated with the different Casimir operators are conserved. They can be summarized as follows:

$$\begin{array}{ccccccccc}
\mathrm{U}(6) & \supset & \mathrm{U}(5) & \supset & \mathrm{SO}(5) & \supset & \mathrm{SO}(3) & \supset & \mathrm{SO}(2) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
[N] & & n_d & & \tau & & \nu_\Delta L & & M_L
\end{array}, \tag{27}$$

$$\begin{array}{ccccccc}
\mathrm{U}(6) & \supset & \mathrm{SU}(3) & \supset & \mathrm{SO}(3) & \supset & \mathrm{SO}(2) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
[N] & & (\lambda,\mu) & & K_L L & & M_L
\end{array},$$

$$\begin{array}{ccccccccc}
\mathrm{U}(6) & \supset & \mathrm{SO}(6) & \supset & \mathrm{SO}(5) & \supset & \mathrm{SO}(3) & \supset & \mathrm{SO}(2) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
[N] & & \sigma & & \tau & & \nu_\Delta L & & M_L
\end{array}.$$

Furthermore, for each of the three hamiltonians in eq. (26) an analytic eigenvalue expression is available,

$$\begin{aligned}
&\mathrm{U}(5): E(n_d, v, L) = \kappa_1 n_d + \kappa_1' n_d(n_d+4) + \kappa_4 \tau(\tau+3) + \kappa_5 L(L+1), \\
&\mathrm{SU}(3): E(\lambda,\mu,L) = \kappa_2(\lambda^2+\mu^2+\lambda\mu+3\lambda+3\mu) + \kappa_5 L(L+1), \\
&\mathrm{SO}(6): E(\sigma,\tau,L) = \kappa_3 \sigma(\sigma+4) + \kappa_4 \tau(\tau+3) + \kappa_5 L(L+1).
\end{aligned} \tag{28}$$

One can add Casimir operators of U(6) to the hamiltonians in eq. (25) without breaking any of the symmetries. For a given nucleus they reduce to a constant contribution. They can be omitted if one is only interested in the spectrum of a single nucleus but they should be introduced if one calculates binding energies. Note that none of the hamiltonians in eq. (26) contains a Casimir operator of SO(2). This interaction breaks the SO(3) symmetry (lifts the M_L degeneracy) and would only be appropriate if the nucleus is placed in an external electric or magnetic field.

The dynamical symmetries of the IBM arise if combinations of certain coefficients κ_i in the hamiltonian (25) vanish. The converse, however, cannot be said: Even if all parameters κ_i are non-zero, the hamiltonian $\hat{H}_{1+2}$ still may exhibit a dynamical symmetry and be analytically solvable. This is a consequence of the existence of unitary transformations which preserve the eigenspectrum of the hamiltonian $\hat{H}_{1+2}$ (and hence its analyticity properties) and which can be represented as transformations in the parameter

space $\{\kappa_i\}$. A *systematic* procedure exists for finding such transformations or parameter symmetries [34] which can, in fact, be applied to any hamiltonian describing a system of interacting bosons and/or fermions.

While a numerical solution of the shell-model eigenvalue problem in general rapidly becomes impossible with increasing particle number, the corresponding problem in the IBM with s and d bosons remains tractable at all times, requiring the diagonalization of matrices with dimension of the order of $\sim 10^2$. One of the main reasons for the success of the IBM is that it provides a workable, albeit approximate, scheme which allows a description of transitional nuclei with a few relevant parameters. Numerous papers have been published on such transitional calculations. We limit ourselves here to citing those that first treated the transitions between the three limits of the IBM: from U(5) to SU(3) [35], from SO(6) to SU(3) [36] and from U(5) to SO(6) [37].

4˙3. *Partial dynamical symmetries.* – As argued in sect. **2**, a dynamical symmetry can be viewed as a generalization and refinement of the concept of symmetry. Its basic paradigm is to write a hamiltonian in terms of Casimir operators of a set of nested algebras. Its hallmarks are (i) solvability of the complete spectrum, (ii) existence of exact quantum numbers for all eigenstates and (iii) pre-determined structure of the eigenfunctions, independent of the parameters in the hamiltonian. A further enlargement of these ideas is obtained by means of the concept of **partial dynamical symmetry**. The idea is to relax the conditions of *complete* solvability and this can be done in essentially two different ways:

1. *Some of the eigenstates keep all of the quantum numbers.* In this case the properties of solvability, good quantum numbers, and symmetry-dictated structure are fulfilled exactly, but only by a subset of eigenstates [38, 39].

2. *All eigenstates keep some of the quantum numbers.* In this case none of the eigenstates is solvable, yet some quantum numbers (of the conserved symmetries) are retained. In general, this type of partial dynamical symmetry arises if the hamiltonian preserves some of the quantum numbers in a dynamical-symmetry classification while breaking others [40, 41].

Combinations of 1 and 2 are possible as well, for example, if some of the eigenstates keep some of the quantum numbers [42].

We emphasize that dynamical symmetry, be it partial or not, is a notion that is not restricted to a specific model but can be applied to any quantal system consisting of interacting particles. Quantum hamiltonians with a partial dynamical symmetry can be constructed with general techniques and their existence is closely related to the order of the interaction among the particles. Applications of these concepts continue to be explored in all fields of physics.

4˙4. *Microscopy.* – The connection with the shell model arises by identifying the s and d bosons with correlated (or Cooper) pairs formed by two nucleons in the valence shell coupled to angular momentum $J = 0$ and $J = 2$. There exists a rich and varied

literature on general procedures to carry out boson mappings in which pairs of fermions are represented as bosons. They fall into two distinct classes. In the first one establishes a correspondence between boson and fermion operators by requiring them to have the same algebraic structure, that is, the same commutation relations. In the second class the correspondence is established rather between state vectors in both spaces. In each case further subclasses exist that differ in their technicalities (*e.g.*, the nature of the operator expansion or the hierarchy in the state correspondence). In the specific example at hand, namely the mapping between the IBM and the shell model, the most successful procedure arguably has been the so-called OAI mapping [43] which associates vectors based on a seniority [U(5)] hierarchy in fermion (boson) space. It has been used in highly complex situations that go well beyond the simple version of IBM-1 with just identical s and d bosons and which include, for example, neutron–proton $T = 1$ and $T = 0$ pairs [44, 45]. In a similar vein a miscroscopic foundation has been given to the IBFM; examples of various mapping techniques for odd-mass nuclei can be found in refs. [46, 47, 48, 49, 50].

4˙5. *The classical limit*. – The coherent-state formalism [51, 52, 53] represents a bridge between algebraic and geometric nuclear models. The central outcome of the formalism is that for any IBM-1 hamiltonian a corresponding potential $V(\beta, \gamma)$ can be constructed where β and γ parametrize the intrinsic quadrupole deformation of the nucleus [54]. This procedure is known as the classical limit of the IBM.

The coherent states used for obtaining the classical limit of the IBM are of the form

$$|N; \alpha_\mu\rangle \propto \left(s^\dagger + \sum_\mu \alpha_\mu d^\dagger_\mu \right)^N |\mathrm{o}\rangle, \tag{29}$$

where $|\mathrm{o}\rangle$ is the boson vacuum and α_μ are five complex variables. These have the interpretation of (quadrupole) shape variables and their associated conjugate momenta. If one limits oneself to static problems, the α_μ can be taken as real; they specify a shape and are analogous to the shape variables of the droplet model of the nucleus [54]. The α_μ can be related to three Euler angles which define the orientation of an intrinsic frame of reference, and two intrinsic shape variables, β and γ, that parametrize quadrupole vibrations of the nuclear surface around an equilibrium shape. In terms of the latter variables, the coherent state (29) is rewritten as

$$|N; \beta\gamma\rangle \propto \left(s^\dagger + \beta \left[\cos\gamma d^\dagger_0 + \sqrt{\frac{1}{2}} \sin\gamma (d^\dagger_{-2} + d^\dagger_{+2}) \right] \right)^N |\mathrm{o}\rangle. \tag{30}$$

The expectation value of the hamiltonian (21) in this state can be determined by elementary methods [55] and yields a functional expression in β and γ which is identified with a potential $V(\beta, \gamma)$, familiar from the geometric model. In this way the following

classical limit of the hamiltonian (21) is found:

$$V(\beta,\gamma) = E_0 + \sum_{n\geq 1} \frac{N(N-1)\cdots(N-n+1)}{(1+\beta^2)^n} \sum_{kl} a_{kl}^{(n)} \beta^{2k+3l} \cos^l 3\gamma, \tag{31}$$

where the non-zero coefficients $a_{kl}^{(n)}$ of order $n = 1$, 2 and 3 are given by

$$\begin{aligned}
a_{00}^{(1)} &= \epsilon_s, \quad a_{10}^{(1)} = \epsilon_d, \\
a_{00}^{(2)} &= \frac{1}{2} v^0_{ssss}, \quad a_{10}^{(2)} = \sqrt{\frac{1}{5}} v^0_{ssdd} + v^2_{sdsd}, \quad a_{01}^{(2)} = -\frac{2}{\sqrt{7}} v^2_{sddd}, \\
a_{20}^{(2)} &= \frac{1}{10} v^0_{dddd} + \frac{1}{7} v^2_{dddd} + \frac{9}{35} v^4_{dddd}, \\
a_{00}^{(3)} &= \frac{1}{6} v^0_{ssssss}, \quad a_{10}^{(3)} = \sqrt{\frac{1}{15}} v^0_{ssssdd} + \frac{1}{2} v^2_{ssdssd}, \\
a_{01}^{(3)} &= -\frac{1}{3}\sqrt{\frac{2}{35}} v^0_{sssddd} - \sqrt{\frac{2}{7}} v^2_{ssdsdd}, \\
a_{20}^{(3)} &= \frac{1}{10} v^0_{sddsdd} + \sqrt{\frac{1}{7}} v^2_{ssdddd} + \frac{1}{7} v^2_{sddsdd} + \frac{9}{35} v^4_{sddsdd}, \\
a_{11}^{(3)} &= -\frac{1}{5}\sqrt{\frac{2}{21}} v^0_{sddddd} - \frac{\sqrt{2}}{7} v^2_{sddddd} - \frac{18}{35}\sqrt{\frac{2}{11}} v^4_{sddddd}, \\
a_{30}^{(3)} &= \frac{1}{14} v^2_{dddddd} + \frac{1}{30} v^3_{dddddd} + \frac{3}{154} v^4_{dddddd} + \frac{7}{165} v^6_{dddddd}, \\
a_{02}^{(3)} &= \frac{1}{105} v^0_{dddddd} - \frac{1}{30} v^3_{dddddd} + \frac{3}{110} v^4_{dddddd} - \frac{4}{1155} v^6_{dddddd},
\end{aligned} \tag{32}$$

in terms of the single boson energies ϵ_s and ϵ_d, and the matrix elements between normalized two- and three-body states,

$$\begin{aligned}
v^L_{\ell_1\ell_2\ell'_1\ell'_2} &= \langle \ell_1\ell_2; LM|\hat{H}_2|\ell'_1\ell'_2; LM\rangle, \\
v^L_{\ell_1\ell_2\ell_3\ell'_1\ell'_2\ell'_3} &= \langle \ell_1\ell_2\ell_3; LM|\hat{H}_3|\ell'_1\ell'_2\ell'_3; LM\rangle.
\end{aligned}$$

The expressions (31,32) are useful for choosing between the many possible three-body interactions.

A **catastrophe** analysis [56] of the potential surfaces in (β,γ) as a function of the hamiltonian parameters determines the stability properties of these shapes. This analysis was carried out for the general IBM hamiltonian with up to two-body interactions by López–Moreno and Castaños [57]. The results of this study are confirmed if a simplified IBM hamiltonian is considered of the form [58]

$$\hat{H}^{\rm ecqf}_{1+2} = \epsilon\, \hat{n}_d + \kappa\, \hat{Q}\cdot\hat{Q}. \tag{33}$$

This hamiltonian provides a simple parametrization of the essential features of nuclear structural evolution in terms of a vibrational term $\hat{n}_d$ (the number of d bosons) and a quadrupole interaction $\hat{Q} \cdot \hat{Q}$ with

$$\hat{Q}_\mu = [s^\dagger \times \tilde{d} + d^\dagger \times s]^{(2)}_\mu + \chi [d^\dagger \times \tilde{d}]^{(2)}_\mu. \tag{34}$$

Besides an overall energy scale, the spectrum of the hamiltonian (33) is determined by two parameters: the ratio ϵ/κ and χ. The three limits of the IBM are obtained with an appropriate choice of parameters: U(5) if $\kappa = 0$, $\mathrm{SU}_\pm(3)$ if $\epsilon = 0$ and $\chi = \pm\sqrt{7}/2$, and SO(6) if $\epsilon = 0$ and $\chi = 0$. One may thus represent the parameter space of the simplified IBM hamiltonian (33) on a triangle with vertices that correspond to the three limits U(5), SU(3) and SO(6), and where arbitrary points correspond to specific values of ϵ/κ and χ. Since there are two possible choices for SU(3), $\chi = -\sqrt{7}/2$ and $\chi = +\sqrt{7}/2$, the triangle can be extended to cover both cases by allowing χ to take negative as well as positive values.

The geometric interpretation of any IBM hamiltonian on the triangle can now be found from its expectation value in the coherent state (30) which for the particular hamiltonian (33) gives

$$\begin{aligned} V(\beta,\gamma) = \frac{N\epsilon\beta^2}{1+\beta^2} + \kappa \Bigg[& \frac{N(5+(1+\chi^2)\beta^2)}{1+\beta^2} \\ & + \frac{N(N-1)}{(1+\beta^2)^2} \left(\frac{2}{7}\chi^2\beta^4 - 4\sqrt{\frac{2}{7}}\chi\beta^3\cos 3\gamma + 4\beta^2 \right) \Bigg]. \end{aligned} \tag{35}$$

The catastrophe analysis of this surface is summarized with the phase diagram shown in fig. 1. Analytically solvable limits are indicated by the dots. Two different SU(3) limits occur corresponding to two possible choices of the quadrupole operator, $\chi = \pm\sqrt{7}/2$. Close to the U(5) vertex, the IBM hamiltonian has a vibrational-like spectrum. Towards the SU(3) and SO(6) vertices, it acquires rotational-like characteristics. This is confirmed by a study of the character of the potential surface in β and γ associated with each point of the triangle. In the region around U(5), corresponding to large ϵ/κ ratios, the minimum of the potential is at $\beta = 0$. On the other hand, close to the $\mathrm{SU}_+(3)$–SO(6)–$\mathrm{SU}_-(3)$ axis the IBM hamiltonian corresponds to a potential with a deformed minimum between $\beta = 0$ and $\beta = \sqrt{2}$. Furthermore, in the region around prolate $\mathrm{SU}_-(3)$ ($\chi < 0$) the minimum occurs for $\gamma = 0^\mathrm{o}$ while around oblate $\mathrm{SU}_+(3)$ ($\chi > 0$) it does for $\gamma = 60^\mathrm{o}$. In this way the picture emerges that the IBM parameter space can be divided into three regions according to the character of the associated potential having (I) a spherical minimum, (II) a prolate deformed minimum or (III) an oblate deformed minimum. The boundaries between the different regions (the so-called Maxwell set) are indicated by the dashed lines in fig. 1 and meet in a triple point. The spherical–deformed border region displays another interesting phenomenon. Since the *absolute* minimum of the potential must be either spherical, or prolate or oblate deformed, its character uniquely determines the three regions and the dividing Maxwell lines. Nevertheless, this does not exclude the

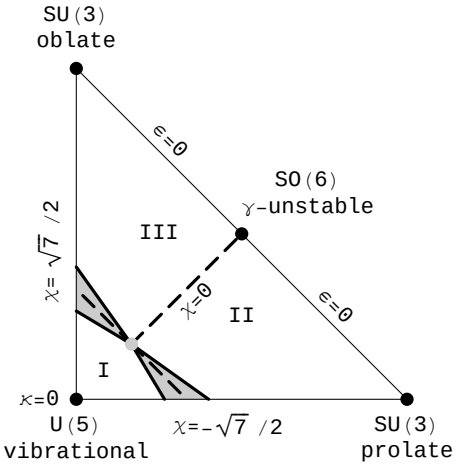

Fig. 1. – Phase diagram of the hamiltonian (33) and the associated geometric interpretation. The parameter space is divided into three regions depending on whether the corresponding potential has (I) a spherical, (II) a prolate deformed or (III) an oblate deformed absolute minimum. These regions are separated by dashed lines and meet in a triple point (grey dot). The shaded area corresponds to a region of coexistence of a spherical and a deformed minimum. Also indicated are the points on the triangle (black dots) which correspond to the dynamical-symmetry limits of the hamiltonian (33) and the choice of parameters ϵ, κ and χ for specific points or lines of the diagram.

possibility that, in passing from one region to another, the potential may display a second *local* minimum. This indeed happens for the U(5)–SU(3) transition [59] where there is a narrow region of coexistence of a spherical and a deformed minimum, indicated by the shaded area in fig. 1. Since, at the borders of this region of coexistence, the potential undergoes a *qualitative* change of character, the boundaries are genuine critical lines of the potential surface [56].

Although these geometric results have been obtained with reference to the simplified hamiltonian (33) and its associated 'triangular' parameter space, it must be emphasized that they remain valid for the general IBM hamiltonian with up to two-body interactions [57].

5. – Triaxiality in the interacting boson model

In this section a first application of the IBM is discussed, namely the use of higher-order interactions between the d bosons and its relation to triaxiality. First, a simplified IBM hamiltonian with up to two-body interactions is described which has been used in the systematic analysis of the collective properties of many nuclei. Although generally yielding satisfactory results when compared to available spectroscopic data, systematic

deviations are observed for properties that are related to (rigid or soft) triaxial nuclear behaviour. It is then argued that such observed deficiencies call for the introduction of three-body interactions between the bosons. Such interactions can be applied to a variety of SO(6)-like nuclei and the example of neutron-rich ruthenium isotopes is presented here. The study described in this section is based on the paper by Stefanescu *et al.* [60] but employs a modified numerical procedure as introduced in ref. [61].

5˙1. *A specific two-body hamiltonian.* – From a great number of standard IBM-1 studies [2] one has a good idea of a workable hamiltonian with up to two-body interactions which is of the form

$$\hat{H}'^{\mathrm{ecqf}}_{1+2} = \epsilon\,\hat{n}_d + \kappa\,\hat{Q}\cdot\hat{Q} + \kappa'\hat{L}\cdot\hat{L} + \lambda\,\hat{n}_d^2, \tag{36}$$

where $\hat{Q}$ is the quadrupole operator (34) and $\hat{L}$ is the angular momentum operator, $\hat{L}_\mu = \sqrt{10}\,[d^\dagger \times \tilde{d}]^{(1)}_\mu$. The $\hat{Q}^2$ and $\hat{L}^2$ terms in eq. (36) constitute the hamiltonian of the so-called consistent-Q formalism (CQF) [62]. Its eigenfunctions are fully determined by χ which for $\chi = \pm\sqrt{7}/2$ gives rise to the deformed or SU(3) limit and for $\chi = 0$ to the γ-unstable or SO(6) limit. In an extended consistent-Q formalism (ECQF) [63] a further term $\epsilon\,\hat{n}_d$ is added with which the third, vibrational or U(5) limit of the IBM-1 can be obtained. The ECQF hamiltonian thus allows one to reach all three limits of the model with four parameters. In some nuclei an additional term $\lambda\,\hat{n}_d^2$ further improves the description of the excitation spectrum. The effect of this term with $\lambda < 0$ is an increase of the moment of inertia with increasing angular momentum (or d-boson seniority τ). This so-called 'τ-compression' has been used for the first time in ref. [64].

For the calculation of electric quadrupole properties an E2 transition operator is needed. In the IBM-1 it is defined as $\hat{T}_\mu(\mathrm{E2}) = e_\mathrm{b}\hat{Q}_\mu$ where e_b is an effective charge for the bosons. In CQF the quadrupole operators in the E2 operator and in the hamiltonian are the same [62], that is, they contain the same χ.

5˙2. *A specific three-body hamiltonian.* – Many nuclear properties can be correctly described by the simple hamiltonian (36) but some cannot. A notable example is the even–odd staggering in the quasi-γ band of nuclei that are close to the SO(6) limit. A characteristic feature of the γ-unstable limit of IBM-1 is a bunching of quasi-γ-band states according to 2^+, $(3^+, 4^+)$, $(5^+, 6^+)$,..., that is, 3^+ and 4^+ are close in energy, *etc.* This even–odd staggering is observed in certain SO(6) nuclei but not in all and in some it is, in fact, replaced by the opposite bunching $(2^+, 3^+)$, $(4^+, 5^+)$,..., which is typical of a rigid triaxial rotor [65]. From these qualitative observations it is clear that the even–odd quasi-γ-band staggering is governed by the γ degree of freedom (*i.e.*, triaxiality) as it changes character in the transition from a γ-soft vibrator to a rigid triaxial rotor.

A proper description of triaxiality in the IBM-1 must necessarily involve higher-order interactions as can be shown from the expressions given in sect. **4**˙5. The minimum of the potential $V(\beta, \gamma)$ in eq. (31) (which can be thought of as the equilibrium shape of the nucleus) of an IBM-1 hamiltonian with up to two-body interactions is either spherical

Table II. – *Normalization coefficients $N_{\lambda L}$ for three-d-boson states.*

L	0	2	3	4	6
$\lambda=0$	—	$\sqrt{\frac{5}{14}}$	—	—	—
$\lambda=2$	$\sqrt{\frac{1}{6}}$	$\sqrt{\frac{7}{8}}$	$\sqrt{\frac{7}{30}}$	$\sqrt{\frac{7}{22}}$	—
$\lambda=4$	—	$\sqrt{\frac{35}{72}}$	$-\sqrt{\frac{7}{12}}$	$\sqrt{\frac{7}{20}}$	$\sqrt{\frac{1}{6}}$

($\beta=0$), prolate deformed ($\beta>0,\gamma=0°$) or oblate deformed ($\beta>0,\gamma=60°$). The lowest term in eq. (31) with a triaxial extremum is quadratic in $\cos 3\gamma$ ($l=2$) and this requires a non-zero $a_{02}^{(3)}$ coefficient. From the explicit expressions given in eqs. (32) it is seen that the lowest-order interactions possibly leading to a triaxial minimum in $V(\beta,\gamma)$ are thus necessarily of the form

$$\hat{H}_3^d=\sum_L \tilde{v}^L_{dddddd}[[d^\dagger\times d^\dagger]^{(\lambda)}\times d^\dagger]^{(L)}\cdot[[\tilde{d}\times\tilde{d}]^{(\lambda')}\times\tilde{d}]^{(L)}, \tag{37}$$

where the allowed angular momenta are $L=0,2,3,4,6$. For several L more than one combination of intermediate angular momenta λ and λ' is possible; these do not give rise to independent terms but differ by a scale factor. To avoid the confusion caused by this scale factor, we rewrite the hamiltonian (37) as

$$\hat{H}_3^d=\sum_L v^L_{dddddd}\hat{B}^\dagger_L\cdot\tilde{B}_L,\qquad \hat{B}^\dagger_{LM}=N_{\lambda L}[[d^\dagger\times d^\dagger]^{(\lambda)}\times d^\dagger]^{(L)}_M. \tag{38}$$

For simplicity's sake the coefficients v^L_{dddddd} shall be denoted as v_L in the following. The normalization coefficient $N_{\lambda L}$ is defined such that $B_{LM}|d^3;LM\rangle$ yields the vacuum state $|\mathrm{o}\rangle$, where $|d^3;LM\rangle$ is a normalized, symmetric state of three bosons coupled to total angular momentum L and z projection M. The normalization coefficients $N_{\lambda L}$ are given in table II for the different combinations of λ and L. Results are independent of λ provided the appropriate coefficient $N_{\lambda L}$ is used.

While there are good arguments for choosing any of the three-body terms $\hat{B}^\dagger_L\cdot\tilde{B}_L$, it is more difficult to distinguish *a priori* between these five different interactions. From the expression for $a_{02}^{(3)}$ given in eqs. (32) it is seen that the three-body term $\hat{B}^\dagger_L\cdot\tilde{B}_L$ with $L=3$ is proportional to $\sin^2 3\gamma$. It is therefore the interaction which is most effective to create a triaxial minimum in the potential $V(\beta,\gamma)$ and for this reason it has been studied in most detail. The effect of $\hat{B}^\dagger_3\cdot\tilde{B}_3$ on even–odd staggering in the quasi-γ band was demonstrated with numerical calculations [66]. Applications of the $L=3$ three-body term were proposed in ref. [67] for SO(6)-like Xe and Ba isotopes in the mass region around $A=130$, as well as for ^{196}Pt.

Most of the results presented below are obtained with the d-boson cubic interaction with $L=3$ which in general reproduces best the quasi-γ-band properties. The terms

with $L \neq 3$ nevertheless have been systematically investigated and those results will occasionally be referred to in the following.

5˙3. *Numerical procedure*. – To test the effectiveness of the various cubic interactions in reproducing the data in near-SO(6) nuclei, the following fitting procedure has been used. The nuclei considered should have enough known states in the ground-state and quasi-γ bands—preferably up to angular momentum $J^\pi = 10^+$—for the procedure to be meaningful. The first step is to determine the parameters in the standard IBM-1 hamiltonian (36). For an initial choice of χ, the parameters κ and κ' are first determined while keeping ϵ_d and λ_d zero. With (κ, κ') thus found as starting values, a new fit is performed setting ϵ_d free as well, leading to the best values $(\kappa, \kappa', \epsilon_d)$. Finally, this process is repeated by letting also λ_d free, leading to a final set $(\kappa, \kappa', \epsilon_d, \lambda_d)$ for a given χ. The parameter χ cannot be reliably determined from energies but is fixed from E2 transition rates which are calculated in the CQF. If not enough E2 data are available, we take χ from a neighbouring isotope. The entire procedure is repeated for different χ, retaining the value that gives the best agreement with the E2 data. In a last step the importance of the $\hat{B}^\dagger_L \cdot \tilde{B}_L$ terms is tested in a similar way by allowing the variation of all five parameters $(\kappa, \kappa', \epsilon_d, \lambda_d, v_L)$ while keeping χ constant. Since one is particularly interested in the influence of v_L on the even–odd staggering, in this final step this parameter is adjusted to the members of the quasi-γ band only. For reasons of numerical stability the weight given to the ground-state-band members is not exactly zero but small.

The accuracy of the fits can be tested by plotting the signature splitting $S(J)$ of the quasi-γ band given by [68]

$$S(J) = \frac{E(J) - E(J-1)}{E(J) - E(J-2)} \cdot \frac{J(J+1) - (J-1)(J-2)}{J(J+1) - J(J-1)} - 1, \tag{39}$$

which vanishes if there is no even–odd staggering.

5˙4. *Results for the neutron-rich ruthenium isotopes*. – In recent years gamma-ray spectroscopy of fission fragments has significantly improved our knowledge of the structure of medium-mass neutron-rich nuclei. In particular, for the heavy fission products ^{108,110,112}Ru, produced by a ^{252}Cf source and studied with the Gammasphere array, new data have become available [69, 70]. As a result of these studies, many more members of the ground-state and quasi-γ bands are now known and this yields important information on the triaxiality character of these nuclei, as will be shown in this section.

The isotopes ^{108,110,112}Ru were already considered in ref. [60] in the context of the IBM-1 with cubic interactions. The advantage of the method presented in this section is that a consistent one- and two-body IBM-1 hamiltonian is taken to which a three-body term is added without changing the value of χ. In this way any improvement of the description of the quasi-γ-band staggering can be unambiguously attributed to the three-body term. Also, a least-squares fit is performed to the parameters in the hamiltonian according to the procedure outlined in sect. **5**˙3. Although all results are obtained with

Table III. – *Parameters and rms deviation for ruthenium isotopes in units of keV.*

Nucleus	ϵ_d	κ	κ'	λ_d	v_3	χ^*	σ
^{108}Ru	1078	−57.6	12.1	−144.9	—	−0.10	23
	852	−66.8	8.3	−130.7	−13.1	−0.10	45
	732	−74.6	14.0	−157.8	30.5**	−0.10	19
^{110}Ru	1053	−46.1	15.5	−123.7	—	−0.10	39
	873	−56.9	9.9	−108.5	−28.1	−0.10	20
^{112}Ru	837	−45.3	15.2	−116.8	—	−0.10	55
	424	−57.8	7.7	−73.7	−46.8	−0.10	38

*Dimensionless. **Value of the coefficient v_2.

a parameter-search routine, there is no guarantee that the absolute value of the root-mean-square (rms) deviation is obtained but the parameters shown in table III define at least a local minimum.

In spite of the differences in fitting procedure the results obtained here are globally in agreement with those of Stefanescu *et al.* [60]. The main conclusion is that, while the staggering pattern of the quasi-γ band is much improved with the $\hat{B}_3^\dagger \cdot \tilde{B}_3$ interaction in ^{110,112}Ru, this is not the case for ^{108}Ru (see fig. 2). This is also evident from the parameters shown in table III: the rms deviation σ actually increases for ^{108}Ru when the three-body interaction is added to the hamiltonian. This is a consequence of our fit procedure which gives (almost) exclusive weight to the quasi-γ-band members when also v_3 is fitted. This increase in σ illustrates that the quasi-γ-band energies in ^{108}Ru cannot be reproduced by adding $\hat{B}_3^\dagger \cdot \tilde{B}_3$ without destroying the agreement for the ground-state band.

As one goes to the heavier ruthenium isotopes, one notices a distinct evolution of the odd–even staggering pattern (see figs. 3 and 4). Whereas the staggering pattern is essentially consistent with the IBM-1 calculation without cubic interactions in ^{108}Ru, this is no longer the case in the two heavier isotopes. In ^{110}Ru there is very little staggering at all, $S(J) \approx 0$, and in ^{112}Ru the staggering pattern in the data is in fact the reverse of what is obtained without cubic interactions, especially at higher angular momenta. The $\hat{B}_3^\dagger \cdot \tilde{B}_3$ interaction shifts levels with even (odd) angular momentum upwards (downwards) in energy and it does so increasingly with increasing spin. This is exactly what can be observed from the data in ^{110}Ru and ^{112}Ru and this provides a strong phenomenological argument for the use of the $\hat{B}_3^\dagger \cdot \tilde{B}_3$ interaction.

One should appreciate the sensitivity of the signature splitting $S(J)$ to the energies of the γ-band levels. This is illustrated with fig. 5 where the energy spectrum of the nucleus ^{112}Ru is shown. It is indeed visible from the figure that energies of the levels of the γ band are better reproduced if cubic interactions are considered but the improvement is much more tellingly illustrated by the plot of $S(J)$ in fig. 4.

From the plot of the signature splitting we can also 'understand' why the $\hat{B}_3^\dagger \cdot \tilde{B}_3$ interaction fails in ^{108}Ru: the deviations in staggering between the data and the IBM-1

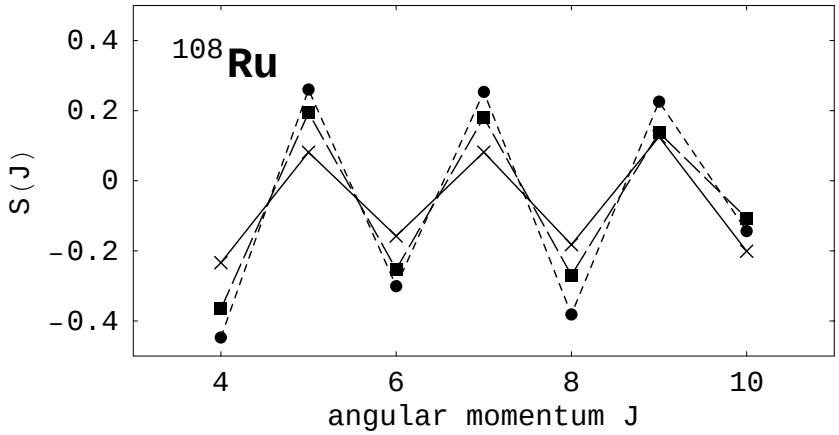

Fig. 2. – Observed and calculated signature splitting for the quasi-γ band in ^{108}Ru. The data are indicated by crosses and the results of the IBM-1 with and without the three-body interaction $\hat{B}_3^\dagger \cdot \tilde{B}_3$ by squares and dots, respectively.

calculation without cubic interactions actually decrease rather than increase with angular momentum. This feature is incompatible with the $L = 3$ term in the hamiltonian (38) but is exactly what is obtained with the $L = 2$ term as shown in fig. 6.

It is important to check that the cubic hamiltonian thus obtained gives reasonable results as regards electric quadrupole transitions. In the initial two-body hamiltonian the E2 transition rates are essentially determined by the value of χ in the quadrupole operator. It is expected that this is still the case when cubic terms are added as long as these do not substantially alter the eigenstates of the hamiltonian. A number of

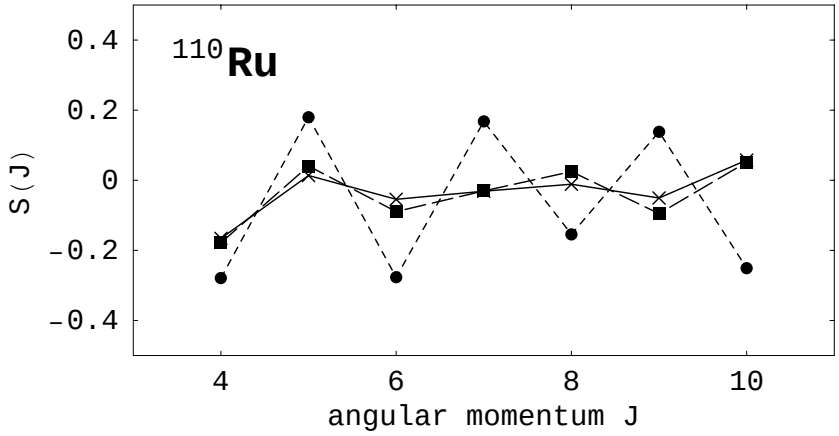

Fig. 3. – Same caption as fig. 2 for ^{110}Ru.

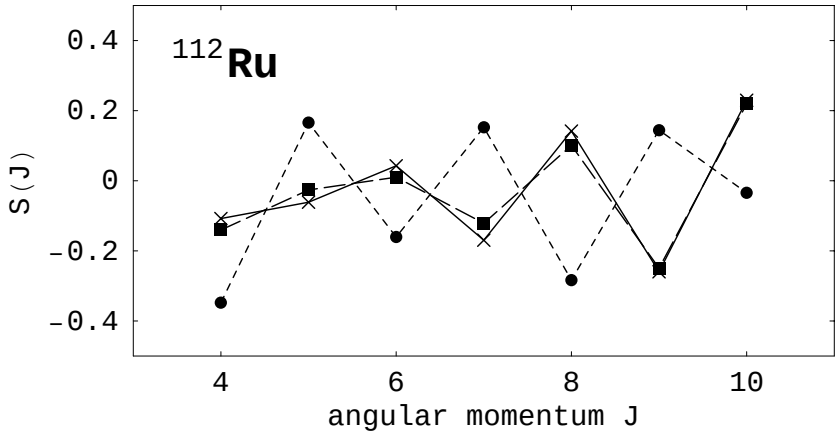

Fig. 4. – Same caption as fig. 2 for ^{112}Ru.

E2 branching ratios from γ-band states are known from recent γ-spectroscopy work on fission products [60] and are compared in table IV with the results of a cubic hamiltonian with the fitted values of v_2 in ^{108}Ru and of v_3 in ^{110,112}Ru in table III. There is a satisfactory overall agreement with the data. The fitted cubic hamiltonian gives

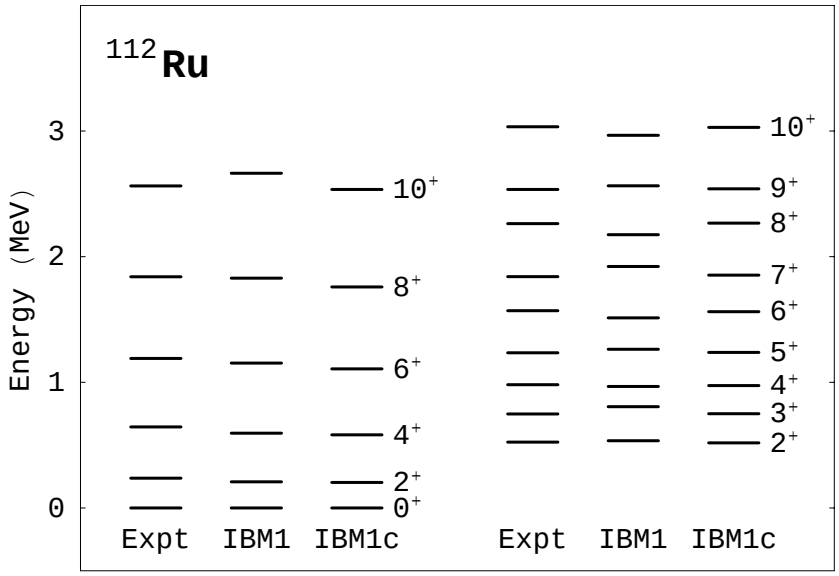

Fig. 5. – Energies of the levels of the ground-state and the γ band up to angular momentum $J^\pi = 10^+$ in the nucleus ^{112}Ru. The three columns correspond to the experimental energies, and the IBM-1 calculation without ('IBM1') and with ('IBM1c') a cubic interaction.

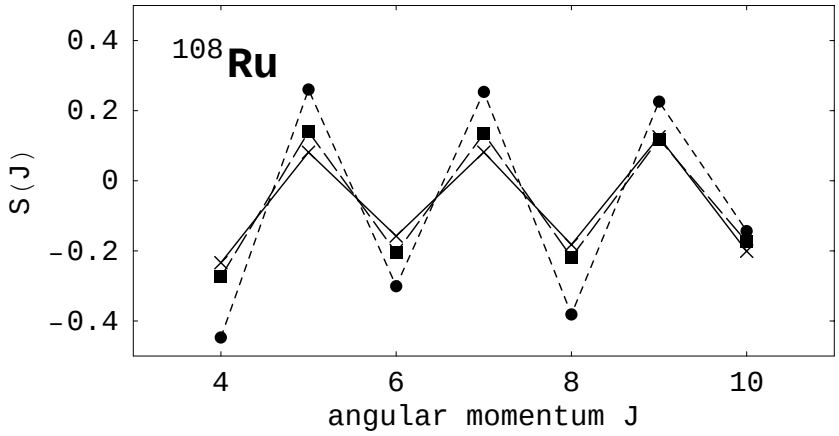

Fig. 6. – Same caption as fig. 2 for the interaction $\hat{B}_2^\dagger \cdot \tilde{B}_2$.

a very large $B(\mathrm{E2}; 4_2^+ \to 2_2^+)/B(\mathrm{E2}; 4_2^+ \to 2_1^+)$ ratio in ^{112}Ru which is due to the accidental vanishing of the $4_2^+ \to 2_1^+$ transition and which 'agrees' with the large value found experimentally. The largest discrepancy between theory and experiment is the $B(\mathrm{E2}; 2_2^+ \to 2_1^+)/B(\mathrm{E2}; 2_2^+ \to 0_1^+)$ ratio which is systematically underpredicted but, on the other hand, the trend of increasing ratio as the mass number increases is correctly obtained in the calculation.

Once the parameters of the hamiltonian have been fitted to the energy spectrum and E2 transition rates, its classical limit yields a potential energy surface $V(\beta, \gamma)$ as obtained from the expression (31). In this way it can be verified to what extent triaxial features are introduced by the cubic interactions. Figure 7 provides an illustration by showing the potential energy surfaces $V(\beta, \gamma)$ for ^{112}Ru obtained in the classical limit of the IBM-1 hamiltonian without and with the $\hat{B}_3^\dagger \cdot \tilde{B}_3$ interaction. The surface on the left-hand side

TABLE IV. – *Experimental and calculated E2 branching ratios for* ^{108,110,112}*Ru.*

	^{108}Ru		^{110}Ru		^{112}Ru	
Ratio	Expt	IBM1c	Expt	IBM1c	Expt	IBM1c
$\frac{2_2^+ \to 2_1^+}{2_2^+ \to 0_1^+}$	8.6(20)	4.6	14.9(2)	10.1	22.2(3)	12.3
$\frac{3_1^+ \to 2_2^+}{3_1^+ \to 2_1^+}$	15.9(14)	11.8	20.4(3)	15.0	21.7(4)	16.2
$\frac{4_2^+ \to 2_2^+}{4_2^+ \to 2_1^+}$	100(5)	53.8	100(6)	373	318(26)	∞
$\frac{4_2^+ \to 2_2^+}{4_2^+ \to 4_1^+}$	2.1(1)	2.6	1.1(1)	1.6	0.94(4)	1.4
$\frac{5_1^+ \to 3_1^+}{5_1^+ \to 4_1^+}$	10(1)	17.6	25(1)	29.3	37(2)	40.8

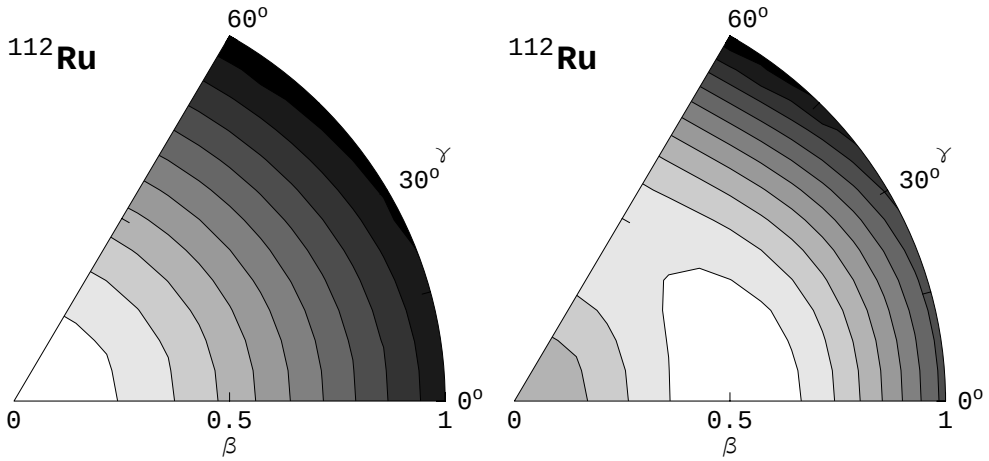

Fig. 7. – Potential energy surfaces $V(\beta,\gamma)$ for ^{112}Ru. The plot on the left-hand side shows the classical limit of the IBM-1 hamiltonian with only two-body interactions while on the right-hand side the effect of $\hat{B}_3^\dagger \cdot \tilde{B}_3$ is included.

is obtained from the two-body hamiltonian and has a minimum at $\beta = 0$ (spherical). The hamiltonian which includes the $\hat{B}_3^\dagger \cdot \tilde{B}_3$ interaction yields the surface on the right-hand side which exhibits a (shallow) minimum at prolate deformation ($\beta \neq 0$ and $\gamma = 0^\circ$). In this example a noticeable change of the potential $V(\beta,\gamma)$ is found as a result of including cubic interactions which perhaps is not surprising since parameter variations are rather important between IBM-1 and IBM-1c in ^{112}Ru (see table III). However, even in this extreme example no triaxial minimum is obtained.

These examples illustrate how a careful analysis of available data may guide the selection of the different interactions in the IBM-1 hamiltonian. While we have currently a good working hamiltonian which includes up to two-body interactions and which describes nuclei throughout the nuclear chart, little is known of the overall trends for three-body interactions. A systematic study of three-body interactions in several series of isotopes where sufficient data are available is currently under way [61]. Although in many nuclei cubic interactions considerably improve the staggering properties of the γ band, in none of the nuclei studied so far a triaxial minimum is obtained and the changes in the potential $V(\beta,\gamma)$ induced by the cubic interactions usually are minor.

6. – Global calculations for spectra and binding energies

If one limits the hamiltonian of the IBM-1 to interactions that are at most of two-body nature between the bosons, the total number of parameters is ten. As shown in table I, six of the parameters determine the energy spectrum of individual nuclei while the four remaining ones exclusively contribute to the binding energy. The parameter systematics of the former is by now well established through phenomenological studies with input

from microscopic theory (for references, see [2]). Surprisingly little has been done with IBM concerning absolute binding energies and in most cases only two-nucleon separation energies have been considered, such as in the recent detailed studies of García–Ramos *et al.* [71] and Fossion *et al.* [72]. The work reported here is most closely related to that of Davis *et al.* [73].

In this section a method based on IBM is proposed that combines spectroscopic information with mass data. For this purpose it is essential to keep in mind that the IBM is a valence-nucleon model which lumps all information on the core of the nucleus into a single constant E_0, its binding energy. The hamiltonian (21) by itself, therefore, cannot provide an adequate description of the *total* binding energy of the nucleus. The method proposed here consists of subtracting a global liquid-drop contribution (which does not include shell or deformation effects) from the nuclear binding energy and modeling the remainder with an IBM-1 hamiltonian. An outline of the method is given as well as a first application of it in the rare-earth region.

Recall that the binding energy $B(N,Z)$ of a nucleus with N neutrons and Z protons is defined through

$$M(N,Z)c^2 = Nm_{\mathrm{n}}c^2 + Zm_{\mathrm{p}}c^2 - B(N,Z), \tag{40}$$

where $M(N,Z)$ is the mass of the nucleus and m_{n} (m_{p}) the mass of the neutron (proton). The binding energy $B(N,Z)$ thus represents the energy needed to pull a nucleus into its $N+Z$ separate nucleons. The binding energy $B(N,Z)$ is positive if the nucleus is bound and energy has to be supplied to pull it apart. Note also that $M(N,Z)$ here refers to the mass of the nucleus only and *not* to that of the atom; so the binding energy $B(N,Z)$ is that of the neutrons and the protons and does not include contributions from the electrons.

A simple, yet surprisingly accurate formula for the binding energy of an atomic nucleus is given by

$$\begin{aligned} B(N,Z) = {} & a_{\mathrm{v}}A - a_{\mathrm{s}}A^{2/3} - a_{\mathrm{c}}\frac{Z(Z-1)}{A^{1/3}} - \frac{S_{\mathrm{v}}}{1+y_{\mathrm{s}}A^{-1/3}}\frac{4T(T+r)}{A} \\ & + a_{\mathrm{p}}\frac{\Delta(N,Z)}{A^{1/2}}, \end{aligned} \tag{41}$$

where $A = N+Z$ is the total number of nucleons. Equation (41) is known as the liquid-drop mass formula. The first three terms appearing in the mass formula are referred to as volume, surface and Coulomb, and have a macroscopic origin that can be understood intuitively by viewing the nucleus as a dense, charged liquid drop. The fourth so-called symmetry term is a consequence of the Pauli principle: Nuclear matter prefers to be symmetric ($N = Z$) because, at constant A, such configuration maximizes availability of the lowest quantum states. The formula (41) uses a somewhat sophisticated form of the symmetry energy where surface and so-called Wigner effects are considered via the inclusion of y_{s} and r, respectively. The last term represents a simple parametrization

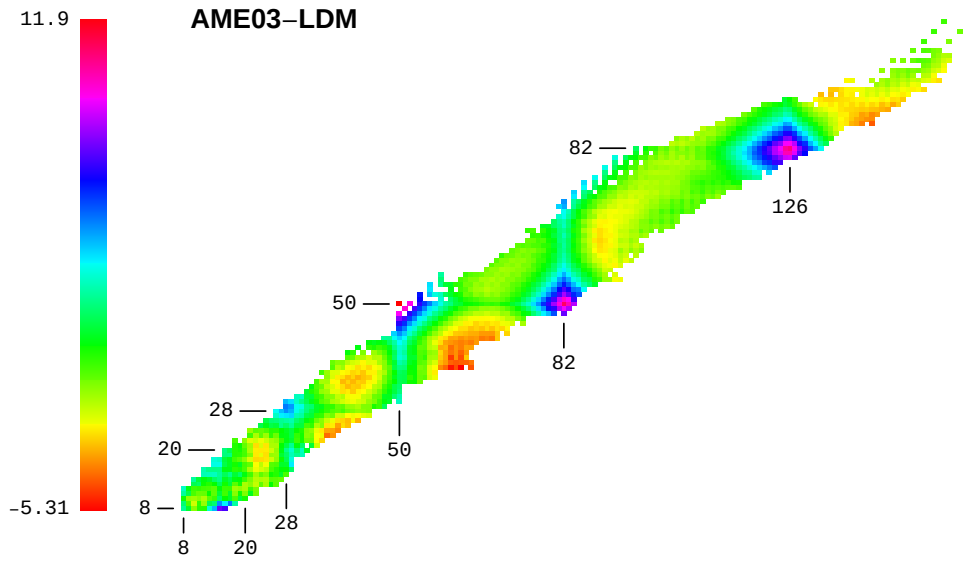

Fig. 8. – Differences between measured and calculated binding energies for nuclei with $N, Z \geq 8$. The binding energies are calculated with the mass formula (41).

of the most important correlation in nuclei, pairing, by assuming $\Delta(N,Z) = +1$, 0 and -1 in even–even, odd-mass and odd–odd nuclei, respectively. In the convention of positive binding energies, the volume and pairing contributions are positive while others are negative; as a result all a coefficients in the formula (41) are positive. Experimental and theoretical progress over the last years has given rise to much more sophisticated mass formulae [74] but the version (41) is sufficient for the present purpose.

In Fig. 8 are shown the differences between the formula (41) with $r = 1$ and the measured nuclear binding energies taken from the 2003 atomic mass evaluation [75]. Immediately obvious from the figure are the large deviations that occur for doubly magic nuclei such as ^{100}Sn, ^{132}Sn or ^{208}Pb which have a diamond-like appearance. This suggests the use of a term linear in $N_\nu + N_\pi$ with N_ρ the number of valence neutron ($\rho = \nu$) or proton ($\rho = \pi$) bosons. Furthermore, the ellipse-like deviations in mid-shell regions suggest another term which is quadratic in $N_\nu + N_\pi$. This simple visual inspection of the deviations thus suggests to add to the liquid-drop mass formula (41) the two-parameter term [76]

$$B_{\rm shell}(N,Z) = a_1(N_\nu + N_\pi) + a_2(N_\nu + N_\pi)^2. \tag{42}$$

This prescription is equivalent to counting valence particles or holes from the nearest closed shell and requires pre-defined magic numbers in the nuclear shell model which are here taken to be $N, Z = 8$, 20, 28, 50, 82, 126 and 184. Note that $N_\nu + N_\pi$ coincides with the total number of bosons which was introduced in sect. **4**; the notation N for

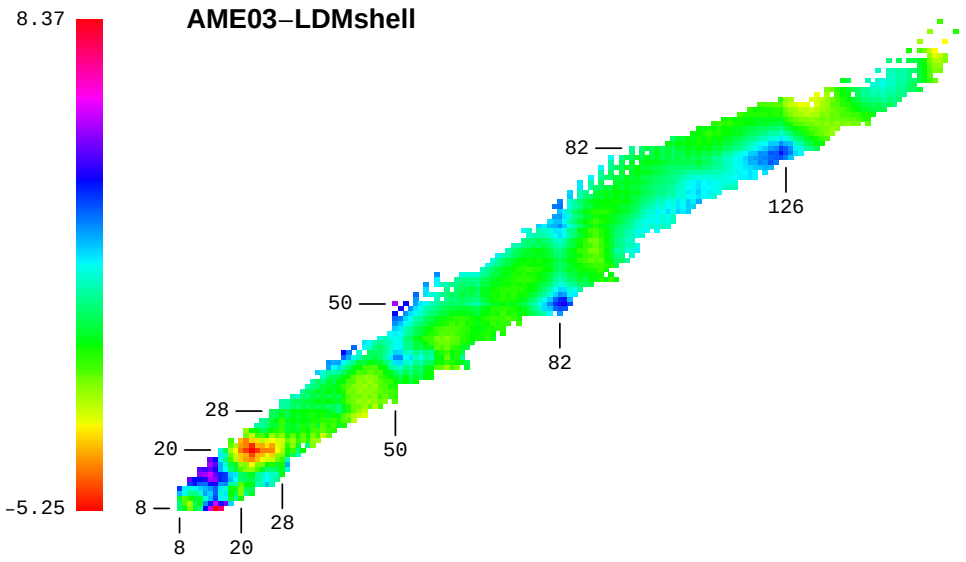

Fig. 9. – Differences between measured and calculated binding energies for nuclei with $N, Z \geq 8$. The binding energies are calculated with the mass formula (41) to which the two-parameter term (42) is added.

this number is not used here in order to avoid confusion with neutron number. The corrections (42) can be considered as a basic version of the successful mass formula of Duflo and Zuker [77].

The use of these two simple corrections reduces the rms deviation for more than 2000 nuclear masses from 2.48 to 1.41 MeV while the values of the macroscopic coefficients remain stable [76]. The shell-corrected plot shown in Fig. 9 has much reduced deviations for the doubly-magic nuclei and in the mid-shell regions of the heavier nuclei. A large fraction of the remaining rms deviation of 1.41 MeV is due to nuclei lighter than ^{56}Ni where shell effects are large and cannot be so easily parametrized.

The significant reduction of the rms deviation with just two terms in the mass formula shows that valence effects are a crucial element in the calculation of nuclear binding energies. This suggests the use of the IBM for mass calculations precisely because it is a valence-nucleon model. Another way of writing the shell correction (42) is in terms of the linear and quadratic Casimir operators of U(6),

$$B_{\rm shell}(N,Z) = a_1' \hat{C}_1[{\rm U}(6)] + a_2' \hat{C}_2[{\rm U}(6)], \tag{43}$$

with $a_1' = a_1 - 5a_2$ and $a_2' = a_2$.

Given the success of the hamiltonian (25) in describing the spectral properties of separate nuclei and of the combination (43) in reducing deviations from a liquid-drop mass formula for many nuclei, one may attempt a description of both properties simultaneously

with the following hamiltonian:

$$\hat{H}^{\rm full}_{1+2} = -B(N,Z) + E_0 + \kappa_0\hat{C}_1[{\rm U}(6)] + \kappa_0'\hat{C}_2[{\rm U}(6)] + \kappa_0''\hat{C}_1[{\rm U}(5)]\hat{C}_1[{\rm U}(6)] + \kappa_1\hat{C}_1[{\rm U}(5)]$$
$$+\kappa_1'\hat{C}_2[{\rm U}(5)] + \kappa_2\hat{C}_2[{\rm SU}(3)] + \kappa_3\hat{C}_2[{\rm SO}(6)] + \kappa_4\hat{C}_2[{\rm SO}(5)] + \kappa_5\hat{C}_2[{\rm SO}(3)]. \quad (44)$$

This expression coincides with the full IBM-1 hamiltonian with up to two-body interactions between the bosons. Note the minus sign in front of $B(N,Z)$ which is needed to convert from positive binding energies to negative absolute energies. The third and fourth terms in eq. (44) are exactly those needed to reduce the rms deviation in the nuclear-mass calculation. The term in κ_0'' represents the product $(N_\nu + N_\pi)\hat{n}_d$ and allows for a d-boson energy which changes with boson number, a feature which is suggested by microscopic theory [43]. The remaining terms coincide with the IBM-1 hamiltonian (25). In summary, the hamiltonian (44) contains all terms up to second order, including a contribution from the core inspired by the liquid-drop model and a one-body term ϵ_d which varies linearly with $N_\nu + N_\pi$. All two-body interactions between the bosons are assumed constant throughout the entire shell; only three-body interactions can represent $(N_\nu + N_\pi)$-dependent two-body interactions.

The hamiltonian (44) can be applied to a set of nuclei belonging to a single major shell which, by way of example, is chosen here to be all even–even nuclei with $82 < N < 126$ and $50 < Z < 82$. Semi-magic nuclei are excluded because they are known to exhibit a seniority spectrum which does not allow an interpretation in terms of IBM. Since a simultaneous fit of many nuclei is attempted with spectra that vary from vibrational to rotational, there exists no obvious *ansatz* for the correct parameter set and an efficient fitting procedure is needed. The method followed here is based on the diagonalization of the error matrix which establishes a hierarchy of the most relevant parameter combinations. The approach is identical to that of the determination of shell-model matrix elements in the *sd* shell [78] and is summarized here for completeness.

The hamiltonian (44) is first written in a simplified notation as

$$\hat{H} = \sum_{i=1}^{P} \kappa_i \hat{O}_i,$$

where κ_i are the P parameters that need to be determined and $\hat{O}_i$ are the P Casimir operators. In the present application the parameters a_i in $B(N,Z)$ have been determined first from a fit to all masses of nuclei with $N, Z \geq 8$. These parameters are kept fixed in the subsequent adjustment of the κ_i to the data set in the shell with $82 < N < 126$ and $50 < Z < 82$. More sophisticated procedures can be envisaged involving iterative or even simultaneous adjustments of a_i and κ_i. If both pieces of the hamiltonian are treated consistently, it will then probably be possible to absorb the constant E_0 into the liquid-drop expression for $B(N,Z)$.

The parameters κ_i are fitted to a data set consisting of M experimental energies $E^k_{\rm expt}$, $k = 1, \ldots, M$. In the shell with $82 < N < 126$ and $50 < Z < 82$, the available data set

comprises 128 ground-state and 1019 excited-state energies. One of the main difficulties in carrying out the present analysis is the selection of relevant data. As the IBM is a model of collective behaviour of nuclei, only excited states of such character should be included, and this selection is far from obvious in many cases. Nevertheless, a selection of this kind has to be carried out and for each selected level a theoretical counterpart is proposed with an energy

$$\lambda_k \equiv \langle \Phi_k | \hat{H} | \Phi_k \rangle = \sum_{i=1}^{P} \kappa_i \langle \Phi_k | \hat{O}_i | \Phi_k \rangle \equiv \sum_{i=1}^{P} \kappa_i \beta_i^k.$$

The wave functions $|\Phi_k\rangle$ are obtained by diagonalizing $\hat{H}$ for an initial choice of parameters $\{\kappa_i^0\}$ and are iteratively improved in the manner explained below.

The optimal set of parameters is obtained by minimization of the rms deviation

$$\chi^2 = \sum_{k=1}^{M} \left(\frac{E_{\text{expt}}^k - \lambda_k}{\sigma_{\text{expt}}^k} \right)^2,$$

where σ_{expt}^k is the error on the experimental energy. Minimization with respect to $\{\kappa_i\}$, under the assumption of κ_i-independent matrix elements β_i^k, leads to a set of linear equations of the form

$$\sum_{i=1}^{P} G_{ij} \kappa_i = e_j, \qquad \text{or} \qquad \kappa_i = \sum_{j=1}^{P} (G^{-1})_{ji} e_j,$$

where $\mathbf{G}$ and $\mathbf{e}$ are $P \times P$ and $P \times 1$ matrices, respectively, defined as

$$G_{ij} = \sum_{k=1}^{M} \frac{\beta_i^k \beta_j^k}{\left(\sigma_{\text{expt}}^k\right)^2}, \qquad e_i = \sum_{k=1}^{M} \frac{E_{\text{expt}}^k \beta_i^k}{\left(\sigma_{\text{expt}}^k\right)^2}.$$

The inverse matrix $\mathbf{G}^{-1}$ is known as the error matrix and contains all information on correlations between parameters. In particular, diagonalization of $\mathbf{G}$ (or $\mathbf{G}^{-1}$) yields a hierarchy of parameters. The diagonalization of $\mathbf{G}$ amounts to finding a unitary transformation $\mathbf{A}$ such that $\mathbf{D} = \mathbf{A}\mathbf{G}\mathbf{A}^{\mathrm{T}}$ is diagonal, $D_{ij} = D_i \delta_{ij}$, or, equivalently, $\mathbf{D}^{-1} = \mathbf{A}\mathbf{G}^{-1}\mathbf{A}^{\mathrm{T}}$ with $(D^{-1})_{ij} = d_i \delta_{ij} = (1/D_i)\delta_{ij}$. The transformation $\mathbf{A}$ defines a set of *uncorrelated* parameters $\nu_i = \sum_j A_{ij} \kappa_j$ with associated errors given by d_i. Consequently, the parameter ν_i can be considered as well determined if the corresponding eigenvalue d_i is small; the ordering of d_i in increasing size thus provides a hierarchy of parameters ν_i. This enables one to use the full hamiltonian (44) with all P Casimir operators but to fit only $p \leq P$ parameter combinations ν_i. For a given number of parameters $p \leq P$ the following fitting procedure can therefore be defined [78]. From an

initial choice of parameters $\{\nu_i^0\}$ a subsequent set is defined according to

$$\nu_i^1 = \begin{cases} \displaystyle\sum_{j=1}^{P} A_{ij}\kappa_j = \sum_{j,j'=1}^{P} A_{ij}(G^{-1})_{j'j}e_{j'}, & \text{if} \quad i \le p, \\ \nu_i^0, & \text{if} \quad i > p, \end{cases}$$

where it is assumed that $\mathbf{A}$ is the unitary matrix which diagonalizes $\mathbf{D}^{-1}$ into eigenvalues d_i that are ordered in increasing value. With this set of parameters $\{\nu_i^1\}$ new wave functions $|\Phi_k\rangle$, matrix elements β_i^k, and matrices $\mathbf{G}$ and $\mathbf{e}$ are obtained with which the next set of parameters $\{\nu_i^2\}$ can be calculated, and so on, until convergence is reached.

Two additional points should be mentioned. The first is that, although ultimately one would like to treat ground and excited states on the same footing, this is not done at present. The *absolute* energies of the 128 ground states are fitted while for excited states the fitted quantity is the excitation energy, that is, the energy *relative* to the ground state. The second point is that the use of the experimental error $\sigma_{\rm expt}^k$ on its own is unsatisfactory since in many cases (*e.g.*, most excitation energies) this error is negligible compared to the rms deviation σ. The proper way to deal with this issue is to consider instead for each experimental data point the error

$$\sqrt{\left(\sigma_{\rm expt}^k\right)^2 + \sigma^2}, \quad \text{with} \quad \sigma^2 = \sum_{k=1}^{M} \left(E_{\rm expt}^k - \lambda_k\right)^2,$$

where σ should be determined iteratively. It is clear that the consideration of the experimental error becomes important only when it is larger than or of the same order as σ. In the present calculation no experimental errors have been taken into account since σ is still relatively large. Strategies for improving the precision of the calculation, as mentioned in the concluding paragraph of this section, might require the consideration of the experimental errors in the future.

Figure 10 shows the rms deviation σ for masses and for excitation energies as a function of the number of parameters up to $p = 10$. In spite of the sophisticated fitting procedure explained in the preceding discussion, convergence towards the optimal parameter set is not guaranteed. In fact, the final parameters, obtained by gradually increasing p starting from $p = 2$, may depend on the choice of the initial set $\{\kappa_i^0\}$. The rms deviations shown in fig. 10, $\sigma_{\rm masses} = 884$ keV and $\sigma_{\rm spectra} = 259$ keV, are those found after a preliminary exploration of the parameter space but they are not necessarily the lowest that can be obtained with the full one- plus two-body IBM-1 hamiltonian.

To summarize this section, a strategy has been outlined for merging the calculations of ground- and excited-state energies and preliminary results have been presented for even–even nuclei in the major shell with $82 < N < 126$ and $50 < Z < 82$. No definitive results for the one- and two-body parameter space are available yet. An further obvious improvement is to include three-body interactions between the bosons which would allow for boson-number-dependent two-body interactions. The overall purpose of the present

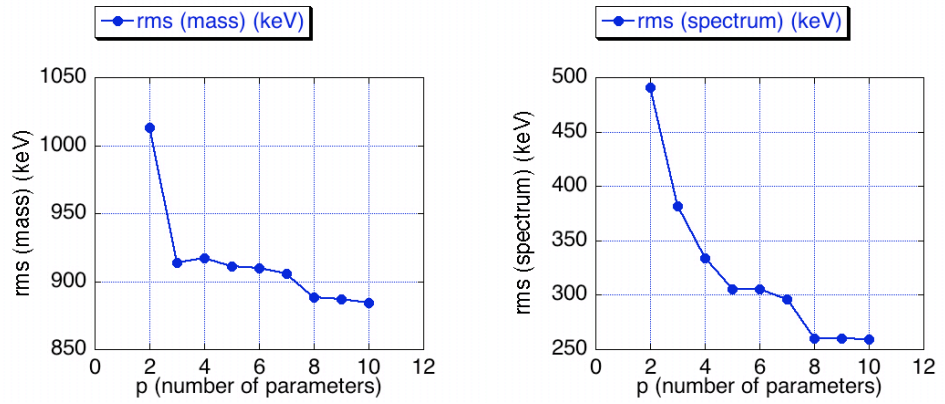

Fig. 10. – The rms deviation σ in units of keV for masses (left) and for excitation energies (right) as a function of the number of parameters p.

approach is that once a reliable parameter set can be determined from known nuclei, it might be of use for the *prediction* of spectral properties of nuclei further removed from the line of stability.

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